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A TREATISE ON
HYDROSTATICS

VOL. I

CONTAINING THE MORE ELEMENTARY
PART OF THE SUBJECT

BY

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PREFACE TO VOL. I

THE present edition of this work is divided into two volumes, the first of which covers the course of hydrostatics required of students who compete for scholarships at the Universities. The book has been, in great part, re-written, and the examples have been very largely increased in number.

Very much of this subject of hydrostatics is easily and profitably treated without the use of the differential and integral calculus—not that the calculus is evaded by artifices more difficult than the principles of the calculus itself. For example, nearly all the practically useful work relating to centres of pressure, and much of that relating to floating bodies, is more easily treated by simple geometry and algebra than by the calculus.

Hence the first volume contains very little of the differential and integral calculus. The fundamental principles of certain forms of turbine have been introduced, as they involve no mathematical difficulties and are of great practical importance.

In the revision of proof-sheets I have had the benefit of the advice of so able and competent a mathematical physicist as Mr. Pidduck of Queen's College.

GEORGE M. MINCHIN.

OXFORD,
September, 1912.

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CHAPTER I

NATURE OF FLUID PRESSURE

1. Experimental Illustration of Pressure. Let a vessel of any shape be fitted with a number of weightless pistons of different areas moving in cylindrical tubes without any friction, and let this vessel be filled with a liquid—suppose water or mercury. We shall suppose also that the piston fittings are perfectly liquid-tight, so that no liquid can escape through the piston tubes.

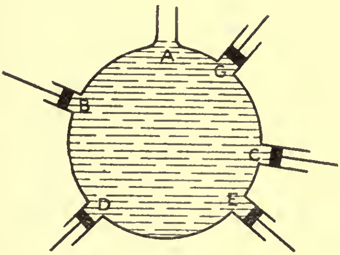


Fig. 1.

Then—especially if the vessel has considerable height and the liquid is mercury—we shall observe that, for the equilibrium of the liquid, each piston requires to be pressed in with a particular force the magnitude of which depends on two things: (1) the area of the piston, and (2) the position of the piston in the vessel.

The forces which urge the pistons out are due, of course, to the weight of the liquid. In the figure A is the highest point of the liquid, and the pistons are fitted at B, C, ... Let P, Q, R, ... be the forces which have to be applied to keep the pistons B, C, D, ... at rest, and let the areas of these pistons be a_1, a_2, a_3, ...; then forces *per unit area* exerted by the liquid on the pistons are

$$\frac{P}{a_1}, \frac{Q}{a_2}, \frac{R}{a_3}, \ldots .$$

These are called the *intensities of pressure* at B, C, D, ...; and in the ideal case which we have supposed (absence of friction, &c.) we should find that the intensity of pressure at a point is greater the greater the *vertical depth* of the point below the highest point, A; and also that the intensity of pressure is the same at any two points whose vertical depths below A are the same. For example, if B and C are in the same horizontal line, $\frac{P}{a_1}$ will be found to be equal to $\frac{Q}{a_2}$, and $\frac{R}{a_3}$ will be greater than $\frac{P}{a_1}$. We shall now vary the experiment. Having supplied to each piston the proper force from without so that there is equilibrium everywhere, let us fit a piston at A, its weight being also negligible. This piston requires, of course, no force to keep it in position. Let s be the area of this piston. Now to any one of the pistons—suppose A—let an additional force, F, be applied to move it in; then we shall find that each of the other pistons will require a special additional force to keep it in position, and we shall find the following simple result: if the intensity of the pressure, $\frac{F}{s}$, applied at A is denoted by f, the additional force required to keep B in position is $f \times a_1$; the additional force required at D is $f \times a_3$, and so on; that is, the intensity of pressure applied at A is transmitted without loss to every other point, B, C, D, G, ...

Precisely the same result is obtained if we produce the additional intensity of pressure, f, by additional force applied at the piston D, or any other piston: this intensity is transmitted, undiminished, by the liquid to every point of the vessel. This result is known as *Pascal's Principle*, which may be thus enunciated: *if intensity of pressure of any amount is applied at any point of the surface of a liquid, the liquid acts as a perfect machine for transmitting this intensity of pressure, unaltered in amount, to all points of the liquid.*

The reason why we began by applying special forces, P, Q, R, ... to the several pistons in order to keep them in position, before applying the intensity of pressure f, is that we desired to eliminate the effect of gravity and to deal with a liquid throughout which no bodily force (such as gravity) acts; and this method enables us to do so as regards points on the surface of the containing vessel; but since the size and shape of the vessel may be altered at pleasure and the result still holds, we can conclude that Pascal's Principle holds for points inside the liquid as well as for points at its surface; but a complete proof of this will be given presently.

If the vessel were filled with a very light gas, such as hydrogen, and it were possible to keep it in without leakage, it would not be necessary to begin by equilibrating the pistons.

Intensity of pressure is a *force per unit area*. Thus if the area of the piston A is ·25 square inches, and the force F applied to the piston is 10 pounds' weight, the intensity of pressure, f, is $\frac{10}{\cdot 25}$, or 40, *pounds' weight* per square inch.

2. Perfect Fluid. A body such that, whatever forces are applied to it, there is no friction between any two particles in contact is called a *perfect fluid*. The action between particles consists wholly of force acting perpendicularly to their surface of contact. Such bodies are, of course, only abstractions, which for many practical purposes, however, may be assumed to exist: water, mercury, gases, and many liquids may be taken as such.

It is usual to divide fluids into *liquids* and *gases*, and to define liquids as fluids that cannot be compressed, or that can be compressed only by applying very great pressure to them; gases are fluids that can be easily compressed.

3. Stress in bodies. If a solid body of any kind is acted upon by any forces, the interior of the body will be strained. Suppose Fig. 2 to represent such a body; and at

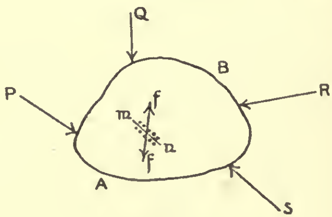


Fig. 2.

any point inside it consider the particles at both sides of a very small plane *mn*. Owing to the forces acting upon the body—of which its weight will be one—the particles at the side *A* of the plane *mn* exert forces on those at the side *B* which are in contact with them. These forces may be of the nature either of *pressure* or of *tension* according to the way in which the forces *P*, *Q*, ... are applied to the body; and the total force exerted on the particles at the *B* side will be a force which we have represented by *f* (acting towards the *B* side). As represented, this force is, on the whole, a *pressure*, but it has a component along the plane *mn*, which is called a *shearing* component. The particles at the *A* side of *mn* experience from those at the *B* side exactly the same force *f* reversed. This force, or *stress*, has, then, two aspects: it acts in one direction or in the opposite according as we consider the action of the *A* side on the *B* side of *mn*, or *vice versa*.

If the body which we are considering is a perfect fluid, this stress f will always be normal to the plane mn, whatever position this plane may have in the fluid. This we shall assume as the definition of a *perfect fluid*.

But in the case of any other kind of body, a great deal depends on the position of the plane *mn*. We may imagine it turned round its centre so as to occupy various positions; and then, as a rule, with each position both the magnitude and the direction of the stress *f* will vary: in some positions of *mn* the stress *f* may lie in the plane, having no normal component. The stress in this case would be pure *shear*. Again, for some positions of *mn* the normal component of *f* may be pressure, and for others tension. In the case of a perfect fluid *f* is always *normal pressure*.

From the fact that fluid stress on a small plane surface acts perpendicularly to the surface we can prove a most important property of this stress—namely, that *its intensity is the same for all small planes at the same point, P, in the fluid*.

Let *P*, Fig. 3, be any point in the fluid (whether liquid or gas); let *cdfb* be a small plane at *P*, and *cdea* be the same plane turned through any angle, 2θ, about the line *cd*; complete the prism by the triangular faces, *abc*, *efd*, perpendicular to the edge *cd*. This prism contains a small volume of the fluid which we can separate in imagination from the rest of the fluid, and *we can regard it as kept in equilibrium*

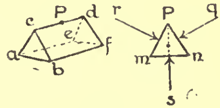


Fig. 3.

by the stresses produced by the surrounding fluid on its faces and by its weight, if gravity is the external force acting on the fluid.

Now, whatever be this external force, its magnitude is proportional to the *volume* of the prism. Let the total stress on the face *cdbf* have a magnitude q: it will act perpendicularly to the face at its middle point. Let the stress on *cdea* be a force r, and that on *abfe* a force s. There will also be stresses acting on the faces *abc* and *efd*, and each of these acts parallel to *cd*.

Let *Pmn* be the cross-section of the prism through P, which point we can take as the middle point of *cd*. Let $cd = l$, $ab = 2c$, $cb = ca = a$; then the areas *cdfb* and *cdea* are each $l\,.\,a$, and the volume of the prism is area $abc \times l$, or $c\,.\,a\cos\theta \times l$. The external force (weight, or other) acting on the prism can be represented by

$$k \times c\,.\,a\,.\,l\cos\theta\,;$$

and if p and p' are the *intensities* of the stresses on the faces *cdfb* and *cdea*, we have

$$q = p \times l\,.\,a, \quad r = p' \times l\,.\,a.$$

Now express the fact that the forces acting on the prism have no component parallel to *mn*, and suppose the external force to make any angle, ϕ, with *mn*; then we have

$$p' \times l\,.\,a\,.\cos\theta - p \times l\,.\,a\,.\cos\theta + k \times c\,.\,a\,.\,l\cos\theta\cos\phi = 0,$$

$$\text{or} \qquad p' - p + k \times c\cos\phi = 0.$$

If now we diminish the size of the prism indefinitely, c vanishes while all the other quantities remain finite, so that the third term, depending on the external force, vanishes, and we have simply

$$p' = p,$$

whatever θ and ϕ may be; that is, the intensity of pressure on the plane *cdfb* is constant whatever be the position of the plane at P; and observe that this *equality* of the pressure intensities depends on the fact that the stress always acts *perpendicularly* to the plane: if q and r did not act normally to the planes on which they act for all positions of these planes, the result would not follow.

4. Equality of Fluid Pressure round a point. The student must avoid the fallacy of supposing that pressure acts *on a point*. Strictly speaking, nothing can act 'on a point': pressure acts *on a surface*; and when we speak of intensity of pressure *at* a point, we mean that pressure acts *on* a small surface placed *at* the point. We see now that if we imagine any small plane area, say 1 square millimetre, placed at a given point P of a fluid, the stress or pressure exerted on this area by the particles at one side of it on those at the other acts normally to the area and has the same magnitude however the position of the little area is varied at that point P. This is the meaning of the expression 'equality of fluid pressure at a point'. No such result holds for any body other than a perfect fluid, as has been already stated.

To each point, then, in a fluid acted on by any forces belongs a special intensity of pressure the amount of which can be calculated if we know the magnitude of the stress, f, exerted on any small plane area, s, at the point, no matter what the position of this area at the point may be: the pressure intensity is

$$\frac{f}{s}.$$

This pressure intensity will, of course, be different for different points in the fluid.

If a fluid is acted upon by no external forces, such as gravity, but only pressure applied somewhere on its surface, the intensity of pressure is the same at all points within it.

Let a perfect fluid be contained within the surface

$ABCD$ (Fig. 4), and suppose pressure to be applied over its surface so that the intensity of this pressure at A is p pounds' weight per square inch. At A take a very small area, s square inches, represented by AA', and on this little area erect a right cylinder, AP, of any length. Now consider the separate equilibrium of the fluid contained within this cylinder. This fluid is held in equilibrium by the force $p.s$ pounds' weight acting on AA', a pressure on the base at P, and a series of pressures all over its curved surface. Resolving forces in the direction AP, we have $p.s =$ the pressure on the base at P, since the pressures on the curved surface are all at right angles to AP. But the area of the base is also s square inches, therefore if $p' =$ intensity of pressure at P,

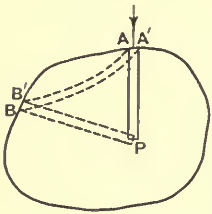


Fig. 4.

$$p.s = p'.s \quad . \quad . \quad . \quad . \qquad (1)$$

$$\therefore \quad p = p'.$$

Again, the pressure intensity at every point on the surface is also p. For, let the base at P be turned round through any angle, and on its new position construct a right cylinder cutting the surface obliquely at B. Let θ be the angle between the normal to the surface at B and the axis, PB, of the cylinder; let p' be the intensity of pressure exerted by the envelope at B on the fluid, and consider the separate equilibrium of the fluid in the cylinder PB. The area of the normal cross-section of the cylinder being s, the area cut off from the surface at B is $s \sec\theta$, and the total pressure on this is $p'.s\sec\theta$. Now resolving along the axis PB for the equilibrium of the enclosed fluid, we have

$$p.s = p'.s\sec\theta.\cos\theta,$$

$$\therefore \quad p' = p.$$

This may also be seen by constructing on the area s at A a tube, $AA'\ BB'$, of any form whatever and of uniform normal cross-section. The fluid inside this tube is kept in equilibrium by the terminal pressures on AA' and BB' together with the pressures of the surrounding fluid which are all normal to the sides of the tube. Hence (except that the terminal forces at A and B are *pressures* and not *tensions*) this fluid is in the same condition as a flexible chain stretched over a smooth surface and acted upon by two terminal forces only, in addition to the continuously distributed normal pressure of the smooth surface; and it is obvious, by considering the equilibrium of each elementary length of the tube, that the forces per unit area at A and B are equal.

Even when the fluid is acted upon by bodily force, such as gravity, the Pascal Principle of the transmission of pressure still holds; that is, any intensity of pressure applied at the surface of the fluid is transmitted undiminished in amount to all parts of the fluid, but in addition to this transmitted pressure there is pressure of different intensities at different points produced by the action of the bodily force.

5. Principle of Separate Equilibrium. The following principle is very largely employed, and has been already used in the previous pages, in the consideration of the equilibrium or motion of a fluid, or, indeed, of any material system:—

We may always consider the equilibrium or motion of any limited portion of a system, apart from the remainder, provided we imagine as applied to it all the forces which are actually exerted on it by the parts imagined to be removed.

Thus, suppose Fig. 5 to represent a fluid, or other mass, at rest under the action of any forces, and let us trace out in imagination any closed surface enclosing a portion, M, of the mass. Then all the portion of the mass outside this surface may be considered as non-existent, so far as M is concerned, if we supply to each element of the surface of M the stress which is actually exerted on it by the mass outside it. The stresses exerted on the elements of surface of M, when the body is a perfect fluid, are represented by the arrows in the figure.

The portion M is, then, in equilibrium under the action of these pressures and whatever external forces (gravity, &c.) also act upon it.

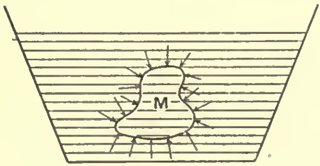


Fig. 5.

Again, it is evident that, having traced out in imagination any surface enclosing a mass, M, of the fluid, we might, without altering anything in the state of this mass M, replace the imagined enclosing surface by an actual material surface, and then remove all the fluid outside this surface; for the enclosing material surface will, by its rigidity, supply to M at each point the pressure which is exerted at that point by the surrounding fluid.

6. The Hydraulic Press. A machine the action of which illustrates the Principle of Pascal is the Hydraulic Press, represented in Fig. 6.

It consists of a stout cylinder, A, in which a cast iron piston, or ram, P, works up and down. This piston has a strong iron platform fixed on the top; on this platform is placed a substance which is to be subjected to great pressure between the platform and a strong plate, D, fixed to four strong vertical pillars. The pressure is applied at the bottom of the piston P by a column of water which is forced into the cylinder A through a tube, t, which communicates with a reservoir of water, B. The water is driven out of B by a force-pump whose piston, p, has a diameter much smaller than that of the piston P; the piston p is worked up and down by means of a lever L, and the cylinder in which p works terminates inside the vessel B in a *rose*, r, the perforations in which admit water while preventing the entrance of foreign matter.

It is easy to see what an enormous multiplication of force can be produced by this machine. If F is the force

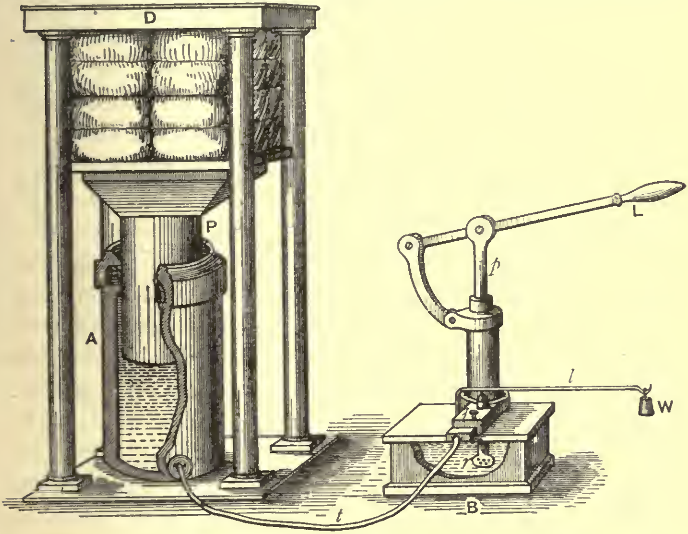


Fig. 6.

applied by the hand to the lever L, n the multiplying ratio of the lever, and s the area of the cross-section of the piston p, the intensity of pressure produced on the water in the vessel B is $\frac{nF}{s}$; so that if S is the area of the cross-section of the piston P, the total force exerted on the end of this piston by the water in A is

$$nF.\frac{S}{s}.$$

Thus, if $S = 100\,s$ and the ratio, n, of the long to the short arm of the lever L is 5, the upward force exerted on the piston P is 500 F, so that if a man exerts a force of 100 pounds' weight on the lever a resistance of nearly 50,000 pounds' weight can be overcome by the piston.

In order to prevent the intensity of pressure in the vessel B from becoming too great, a safety-valve closed by a lever loaded with a given weight, W, is employed.

The Hydraulic Press remained for a long time comparatively useless, because the great pressure to which the water was subject drove the liquid out of the cylinder A between the surface of the piston P and the inner surface of the cylinder. This defect was remedied in a very simple and ingenious manner by Bramah, an English engineer, in the year 1796. In the neck of the cylinder A is cut a circular groove all round, and into this groove is fitted a leather collar the cross-section of which is represented at c in the form ∩. This collar is saturated with oil, in order that it may be water-tight, and it will be seen that it presses with its left-hand and upper portion against the cylinder A, while its right-hand portion is against the piston P. When, by pressure, the water is forced up between the surface of the piston and the surface of the cylinder, this water enters the lower or hollow portion of the inverted U-shaped collar and firmly presses the leather against both the piston P and the surface of the groove, thus preventing any escape of water from the cylinder.

In consequence of this great improvement in the machine, it is very commonly called *Bramah's Press*.

In order to prevent the return of the water from the cylinder A on the upward stroke of the piston p, there is a valve, represented at i in Fig. 6, and shown more clearly at i in Fig. 7, which is a simple sketch of the essentials of the force-pump. When the piston p rises, a valve, e, in the pipe dipping into the reservoir *B*, opens upwards and allows the water to fill the cylinder *J* and to flow through *o* to the valve *i*. When *p* descends, the water closes the valve *e* and is forced to open the valve *i* which is pressed down by a spiral spring. When the piston *p* moves upwards, the water which has passed the valve *i* into the cylinder *A* cannot return into the cylinder *J* because it obviously assists the spring in closing the valve *i*. The safety-valve is represented at *v* in Fig. 7.

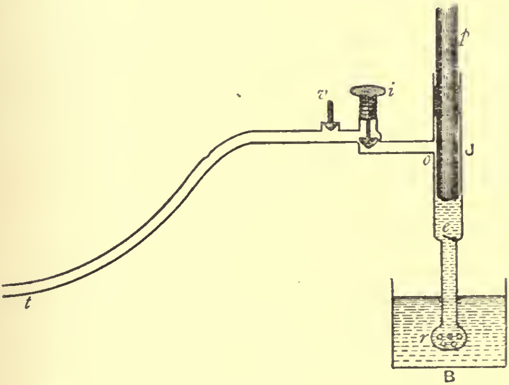


Fig. 7.

The piston *p* works in a stuffing-box in the upper part of the cylinder *J*, this stuffing-box playing the same part as the leather collar round the ram—i. e., preventing leakage. The piston must not fit the lower part of the cylinder *J* tightly, because when *p* in its downward motion passes *o*, the cylinder would be burst if the water above the closed valve *e* could not escape round the piston and out through the valve *i*.

Another machine depending essentially on the same principles and illustrating the Principle of Pascal is the

Hydrostatic Bellows, which is formed by two circular boards connected, in bellows fashion, by water-tight leather, the boards being the ends of a cylinder the curved surface of which is formed by the leather. One of these boards being placed on the ground, the other lies loosely on top of it. A narrow tube communicates with the interior of this cylinder. If this tube is a long one and held vertically, when water is poured into it at its upper end, the upper board of the bellows, and any load that may be placed on it, will be raised by the pressure of the water, the intensity of which pressure depends (as will be subsequently explained) on the height to which the narrow tube is filled.

7. Specific Weight. The specific weight of any substance means its *weight per unit volume*. If w is the weight of a homogeneous body per unit volume, the volume of the body being V and the weight W,

$$W = V . w.$$

It will be useful to remember that

1	cubic foot of water has a mass of about		$62\frac{1}{2}$ lbs.
1	,, ,, ,, ,, ,,	,, ,,	1000 ounces.
1	,, inch of mercury	,, ,,	·491 lbs.

These numbers are, of course, only approximate, because the mass of a cubic foot of any substance depends on the temperature of the substance.

A *gramme* is defined to be the mass, or quantity of matter, in 1 cubic centimètre of water when the water is at its temperature of maximum density; this temperature is very nearly 4° C.

A term in frequent use is the *specific gravity* of a substance, which ought, apparently, to signify the same thing as its specific weight; but it does not. The specific gravity of any homogeneous solid or liquid means, in its ordinary employment, the *ratio* of the weight of any volume of the substance to the weight of an equal volume of distilled water at the temperature 0° C. Thus, for example, in the following table of specific gravities:

gold	19·3
silver	10·5
copper	8·6
platinum	22·0
sea-water	1·026
alcohol	·791
mercury	13·596

the number opposite the name of any substance does not tell us the weight of a cubic foot, or of any other volume, of the substance; it merely tells, with regard to platinum, for example, that a cubic foot of it, or a volume V of it, is 22 times as heavy as a cubic foot, or a volume V, of distilled water. A table of specific gravities is a table of *relative* weights of equal volumes. In the C. G. S. system, since the unit of weight is that of 1 cubic centimètre of water, and since water is the substance with which in a table of specific gravities all solids and liquids are compared, the number (specific gravity) opposite any substance expresses the actual mass, in grammes, of 1 cubic cm. of the substance.

If s is the specific gravity of any substance and w the actual weight of a unit volume of the standard substance (water), the weight of a volume V of the substance is given by the equation

$$W = Vsw.$$

The term *density* is used, as has been said, to denote the *mass per unit volume* of a substance. Thus if mass is measured in grammes and volume in cubic centimètres, the density of silver is 10·5 grammes per cubic centimètre; the density of mercury is 13·596 grammes per cubic cm. If mass is measured in pounds and volume in cubic inches, the density of silver is ·3797 lbs. per cubic inch and that of mercury ·491 lbs. per cubic inch. These latter numbers are, of course, proportional to the former.

The term *density* has no reference to gravitation. If silver and mercury are taken from the Earth to a position in interstellar space in which there is felt no appreciable attraction from any Sun or Planet, it is still true that silver has a mass of 10·5 and mercury a mass of 13·596 grammes per cubic cm. Neither would, in this position, have any specific *weight*, since there is no external force of attraction acting on them; but the moment they are taken to the surface of any planet each acquires weight, and the *ratio* of the weights of equal volumes of them is the ratio, 10·5 : 13·596, of their densities. If, for example, they were carried to the surface of the planet Jupiter, the weight of a cubic cm. of each would be nearly $2\frac{1}{2}$ times as great as it is on the surface of the Earth; but a table of *relative* weights of substances on Jupiter would be exactly the same as a table of relative weights on the Earth.

If any given volumes of a number of homogeneous substances are mixed together in such a way as to make a homogeneous mixture whose volume is the *sum* of the volumes of the separate substances, the specific weight of the mixture is easily found. For, let v_1 and w_1 be the volume and specific weight of the first substance; v_2 and w_2 those of the second; and so on. Then if w is the required specific weight of the mixture, since the weight of the mixture is equal to the sum of the separate weights,

$$(v_1 + v_2 + v_3 + \ldots)\, w = v_1 w_1 + v_2 w_2 + v_3 w_3 + \ldots;$$

$$\therefore\ w = \frac{\Sigma\, vw}{\Sigma\, v}.$$

Such a mixture is called a *mechanical* mixture—as, for instance, a mixture of sand and clay. But when a chemical combination takes place between any of the substances, the volume of the mixture is not equal to the sum of the volumes mixed—as when sulphuric acid is mixed with water. If for any chemical mixture V (which must be specially measured) is the volume of the mixture, it is evident that we have, as above,

$$w = \frac{\Sigma\, vw}{V}.$$

EXAMPLES.

1. A cask A is filled to the volume v with a liquid of specific weight w; another cask, B, is filled, also to the volume v, with another liquid of specific weight s; $\frac{v}{n}$ is taken out of A and $\frac{v}{n}$ also out of B, the first being put into B and the second into A, and the contents of each cask are shaken up so that the liquid in each becomes homogeneous. The same process is repeated again and again: find—

(*a*) the specific weight of the liquid in each cask after m such operations;

(*b*) the volume of the original liquid in each cask.

Result. If $w-s$ is denoted by d, and if w_m, s_m are the specific weights of the liquids in A and B, respectively, after m operations,

$$w_m = w - \frac{d}{2}\left\{1 - \left(1 - \frac{2}{n}\right)^m\right\},$$

$$s_m = s + \frac{d}{2}\left\{1 - \left(1 - \frac{2}{n}\right)^m\right\},$$

and the volume of the original liquid in either cask is

$$\frac{v}{2}\left\{1 + \left(1 - \frac{2}{n}\right)^m\right\}.$$

[N.B. The liquids are assumed not to enter into chemical combination.]

2. In a hydraulic press an effort of 20 pounds' weight is applied

at the end of a lever, at a distance of 9 inches from the fulcrum, moving the plunger of a force-pump, the point of attachment of the plunger being 4 inches from the fulcrum. The diameter of the plunger is 1 inch and that of the ram 8 inches; find the thrust exerted by the ram.

$1\frac{2}{7}$ tons' weight.

3. In a hydraulic press the diameter of the plunger is 1 inch and that of the ram 1 foot; the effort is applied at a distance of 10 inches from the fulcrum, and the point of attachment of the plunger is 5 inches from the fulcrum; calculate the magnitude of the effort required to produce a thrust of 2 tons' weight by means of the ram.

$15\frac{5}{9}$ pounds' weight.

CHAPTER II

THEOREM OF PLANE-MOMENTS

8. If any number of particles whose masses are m_1, m_2, m_3, ... are at distances z_1, z_2, z_3, ... from any plane, it is known that the distance, $\bar{z}$, of their centre of mass, or centre of gravity, from the plane is given by the equation

$$\bar{z} = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}; \quad . \quad . \quad . \quad (a)$$

and the same expression gives the distance of the 'centre' of a system of parallel forces P_1, P_2, P_3, ... from a plane, the distances of their separate points of application from the plane being z_1, z_2, z_3, ... (see *Statics*, vol. i, chap. xi).

In either case if distances measured from one side of the plane are taken as positive, distances measured from the other side are to be taken as negative; and, in addition, if parallel forces acting in one sense are taken as positive, those forces which act in the opposite sense are negative.

The result (a) holds, of course, for particles occupying positions in space of three dimensions, as well as for particles lying in one plane; and similarly for parallel forces.

Owing to the importance of this equation in all calculations, we give a diagrammatic representation of the way in which the student will find it convenient for actual calculation. It is to be observed that no such result as (a) holds for distances measured from an *axis*, or a line; but in many cases the result is applicable, by accident, to distances measured from a line. Thus, if all the particles, or points of application of parallel forces, happen to lie in one plane (P), the above will give the distance of their centre from any line (L) in the plane, but only because the distances $z_1, z_2, \ldots$ measured from (L) are really measured from a plane through (L) perpendicular to (P).

We shall refer to (a) as the *theorem of mass-moments*, or *theorem of plane-moments*.

It is obvious that the mass-moment of any system with respect to any plane passing through its centre of gravity is zero.

It is also evident that the distances $z_1, z_2, \ldots$ need not be perpendiculars; they may be oblique distances—all, of course, measured in the same direction.

The work of practical calculation is often facilitated by forming tables of masses, distances, and products, in columns, as in the following example.

EXAMPLE.

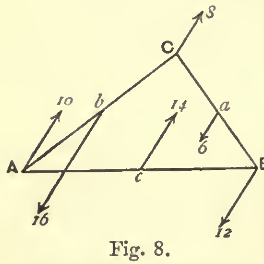


Fig. 8.

At the vertices, A, B, C (Fig. 8) of a triangle and at the middle points, a, b, c, of the opposite sides act parallel forces whose magnitudes and senses are represented in the figure; find the position of the centre of the system.

The position of the centre will be known if its distances from any two sides, AB and AC, are known.

To find its distance from AB, let the length of the perpendicular from C on AB be p; and form a table of forces and distances of their points of application from AB as in the following scheme, taking as positive those forces which act in the sense of that at A:—

Forces.	Distances from AB.	Products.	Distances from AC.	Products.
10	0	0	0	0
−16	$\frac{1}{2}p$	$-8p$	0	0
8	p	$8p$	0	0
−6	$\frac{1}{2}p$	$-3p$	$\frac{1}{2}q$	$-3q$
−12	0	0	q	$-12q$
14	0	0	$\frac{1}{2}q$	$7q$
−2		$-3p$		$-8q$

The sum of the first column answers to Σm, the denominator of (a), p. 19, while the sum of the third column answers to Σmz, the numerator, so that the perpendicular distance of the centre from AB is

$$\frac{-3p}{-2}, \text{ or } \frac{3}{2}p.$$

Drawing, then, a line parallel to AB at a distance $\frac{3}{2}p$ (above C), we know that G lies somewhere on this line.

Denoting the perpendicular from B on AC by q, forming a table (column 4) of distances from AC, and a column (number 5) of corresponding products, and dividing the sum of these products by the sum of the forces, we have the distance of G from AC equal to

$$\frac{-8q}{-2q}, \text{ or } 4q.$$

Hence G lies on a line to the right of B distant $4q$ from AC. The point of intersection of this with the previous line is G.

Thus also to find the position of the 'centre of gravity' of a trapezium $ABCD$ whose parallel sides AB and CD are $2a$, $2b$, and height h, divide it into two triangles by the diagonal AC; then the areas ABC, ADC are ah, bh, and they may be taken as represented by a, b, since the m's in equation (a) may all be multiplied by any common factor. Take mass-moments about

AB; the distances of the centres of gravity from AB are $\frac{h}{3}$, $\frac{2h}{3}$; therefore the diagram is

masses	distances from AB	products
a	⅓h	⅓ah
b	⅔h	⅔bh
a + b		⅓(a+2b)h

Fig. 9.

and dividing the sum of the third column by the sum of the first, we have

$$\bar{z} = \frac{a+2b}{a+b} \cdot \frac{h}{3}.$$

As the centre of gravity lies on the line joining the mid points of AB and CD, it is therefore found.

9. Simple cases of Continuously Distributed Forces. We shall now find the centre of a system of parallel forces distributed continuously over a plane area, in a few simple cases which do not require the application of the Integral Calculus.

(1) If normal pressure acts all over any plane area in such a way that its intensity is the same at all points, the resultant pressure acts at the centre of area ('centre of gravity,' so called) of the figure. For, if at any two points, A, B, in the given figure we take any two elements of area, the pressures on them are directly proportional to the areas themselves, and the resultant of these forces acts at a point in the line AB dividing it into segments inversely as the forces, i. e., inversely as the areas; hence it acts at the centre of area of the two elements of area.

The process, therefore, of finding the point of application of the resultant of the whole system of pressures acting on the indefinitely great number of elements of area into which the given figure can be broken up is precisely the same as that of finding the centre of area of the figure.

(2) If parallel forces act at all points of a right line, AB, Fig. 10, in such a way that the force at any point P is directly proportional to the distance, PA, of P from one extremity, A, of the line, the resultant force acts at the point on AB which is $\frac{2}{3}$ of the length AB from A.

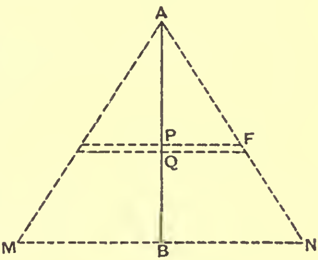


Fig. 10.

For, imagine AB to be broken up into an indefinitely great number of small equal parts, PQ; describe any isosceles triangle, MAN, having AB for height, and from all the points, P, Q, ... of division of AB draw parallels to the base MN, thus dividing the area MAN into an indefinitely great number of narrow strips. The area of any strip, QF, is simply proportional to the distance, PA, of the strip from A. Hence the areas of the strips are exactly proportional to the given system of parallel forces; but the centre of area of the strips, or centre of area of the whole triangle, is $\frac{2}{3}$ AB from A. This point is, then, the centre of the given system of forces.

(3) If parallel forces act at all points of a right line, AB, Fig. 10, in such a way that the force at any point P is proportional to the *square* of the distance, PA, of P from one extremity, A, of the line, the resultant force acts at a point on AB which is $\frac{3}{4}$ of the length AB from A.

For, imagine AB to be broken up, as before, into equal elements, such as PQ; describe any solid cone having AB for its axis, and let this cone be represented by MAN. From all the points of division of AB draw planes parallel to the base MN of the cone, thus dividing the cone into an indefinitely great number of thin circular plates. The volume of the plate at P is $\pi PF^2 \times PQ$, and since the thicknesses of the plates are all equal to PQ, the volume of the plate is proportional to PF^2, i. e., to PA^2. Hence the volumes of the plates vary exactly as the forces of the given system, and therefore the centre of volume of the plates is identical with the centre of the force system; but the former (centre of volume of the cone) is $\frac{3}{4}$ AB from A; therefore, &c.

(4) If parallel forces act at all points of a right line AB, in such a way that the force at any point P is proportional to the product of the distances PA, PB of P from the extremities of the line, the resultant force acts at the middle point of AB.

For, taking a point, P', whose distance from B is equal to that of P from A, the forces at P and P' are evidently equal; their resultant therefore acts at the middle of AB. Hence the system of forces from A to this middle point is the same as the system from B to this point; the resultant, therefore, of the whole system acts at the middle point of AB.

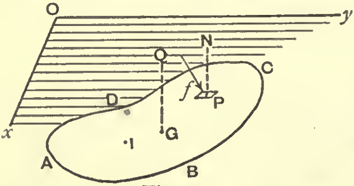


Fig. 11.

(5) If each infinitesimal element of any plane area is acted upon by normal pressure proportional conjointly to the magnitude of the area and to the distance of the element from a given plane, the *magnitude* of the resultant pressure will be proportional to

the product of the whole plane area and the distance of its centre of area from the given plane.

For, let yOx, Fig. 11, represent the given plane, and $ABCD$ the given plane area. Take any point, P, in this area, and round P describe a very small closed curve whose area is s. Let PN, the perpendicular from P on the plane yOx, be denoted by z; then, by hypothesis, the amount of force, f, on the element at P is given by the equation

$$f = k \,.\, sz,$$

where k is a given constant. If s', s'',... are any other elements of area whose distances from the given plane are z', z'',... the resultant pressure, being equal to the sum of f, f', f'', ... of the individual pressures on the elements, is equal to

$$k\,(sz + s'z' + s''z'' + \ldots).$$

But if $A =$ the area of the whole plane figure, and $\bar{z}$ is the distance, GQ, of the centre of area, G, from yOx, we have

$$A\bar{z} = sz + s'z' + s''z'' + \ldots .$$

Hence if P is the resultant pressure,

$$P = k \,.\, A \,.\, \bar{z}. \qquad \ldots\ldots \qquad (a)$$

The student must be careful to observe that the resultant pressure *does not act at* G, but evidently at some such point as I, whose distance from the plane yOx is greater than the distance of G from the plane.

In this case, then, the mean intensity of pressure on the area is that which exists at G.

CHAPTER III

LIQUID PRESSURE ON PLANE SURFACES

10. Intensity of Pressure produced by Gravity. Let *ACB*, Fig. 12, be a vessel of any shape containing water or other homogeneous liquid. Then at each point, *P*, of the liquid the action of gravity produces a certain intensity of pressure, the magnitude of which we proceed to find. At *P* draw an indefinitely small horizontal element of area —*s* square inches, suppose —and on the contour of this area describe a vertical cylinder, *PN*. Consider now the separate equilibrium (Art. 5) of the liquid in this cylinder.

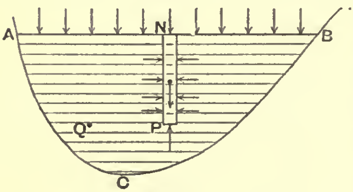


Fig. 12.

If *PN* is z inches in length, the volume of the cylinder $= z . s$ cubic inches, and if the specific weight of the liquid is w pounds' weight per cubic inch, the weight of the cylinder $= wzs$. This cylinder is acted upon by a vertically upward pressure on the base s at *P* and a system of horizontal pressures round its curved surface, in addition to its weight—omitting, for the present, the surface pressure at *N* produced by the atmosphere or any other cause.

If p pounds' weight per square inch is the intensity of pressure at P, the upward pressure on the base s is $p \cdot s$. Resolving forces vertically, we have, then,

$$p \cdot s = wz \cdot s;$$

$$\therefore \quad p = wz, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

which gives the required intensity of pressure.

If the surface intensity of pressure is p_0 pounds' weight per square inch, this will be added to the value (a), by Pascal's principle; hence the complete value of p is given by the equation

$$p = wz + p_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

Observe that we have not assumed the bounding surface AB to be horizontal.

Without any reference to the shape of the surface AB, we can see that the intensity of pressure is the same at all points P, Q, ... which lie in the same horizontal plane.

For, draw PQ; at P and Q place two indefinitely small equal elements of area, s, perpendicularly to PQ; form a cylinder having PQ for axis and these little areas for bases, and consider the separate equilibrium of the liquid enclosed in this cylinder. The forces keeping it in equilibrium are its weight, a system of pressures all round its curved surface, and the pressures on its bases at P and Q. Resolving forces along PQ for equilibrium, neither the weight nor the system of pressures on the curved surface will enter into the equation; therefore the pressure on the base s at $P =$ the pressure on the (equal) base s at Q; that is, the intensity at $P =$ the intensity at Q.

From this it follows that the bounding surface AB on which at all points there is either no pressure, or pressure of constant intensity, *must be a horizontal plane.*

For, take any two points, P, Q, in a horizontal plane, and let their vertical distances below AB be z and z'.

Then by (β), we have

$$wz' + p_0 = wz + p_0;$$
$$\therefore \quad z' = z,$$

that is, all points in the same horizontal plane are at the same depth below the surface AB—which proves AB to be a horizontal plane.

It is usual to speak of the surface, AB, of contact of the liquid with the atmosphere as the *free surface* of the liquid. It is simply a surface at each point of which the intensity of pressure is constant, the constant being the atmospheric intensity.

The result at which we have arrived may be also stated thus—*all points in a heavy homogeneous liquid at which the intensity of pressure is the same lie in a horizontal plane;* and from this it follows that if a mass of water partly enclosed by subterranean rocks, &c., has access to the atmosphere by any number of channels, the level of the water will be the same in all these channels. It is to be observed that z in (α) and (β) *is the depth of the point* P (Fig. 13) *below the free surface*—not the distance, PD, of the point P from the roof of the cavity in which the water is partly confined.

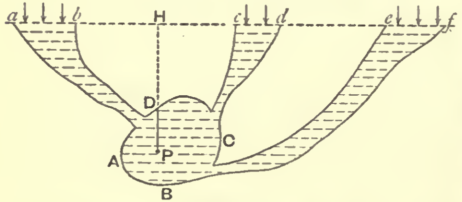


Fig. 13.

We may here, if we please, consider the separate equilibrium of a small vertical cylinder of the liquid terminating at D, and we shall have simply the result (β) in which, however, p_0 would now mean the vertical downward component of the pressure intensity of the roof of the cavity at D on the water. But the result (β) holds for the intensity of pressure at P if PH is put for z, where H is the foot of the perpendicular from P on the plane of the free surfaces *ab*, *cd*, *ef* of the water; for, nowhere in the liquid will the state of affairs be altered if we imagine the roof of the cavity to be removed, and the space bDc to be filled with water up to the level bc. In this way we shall have a vertical cylinder, PH, unobstructed by the roof, and terminating on the free surface.

It is usual to illustrate the fact that all parts of the free surface of a liquid lie in a horizontal plane by taking a vessel, ABC, of any shape and fitting into it tubes or funnels of various forms, and then pouring water in through any one of these tubes, the visible result being that the water stands at the same level in all the tubes. This is, indeed, nothing more than the principle of separate equilibrium (see end of Art. 5); for, these variously shaped funnels may be supposed to have been surfaces traced out in imagination in a large vessel of water whose free surface was *af*, these imagined surfaces being then replaced by material tubes, and the outside liquid removed. The level of the liquid in each tube would still be *af*.

11. Superposed Liquids. If in a vessel, AOB, Fig. 14, several liquids be placed as layers, one on top of another, there being no chemical combination between them, the common surface of each pair of liquids is a horizontal plane. Let the specific weights of the liquids be w_1, w_2, w_3, ... The free surface, AB, has been already proved to be a horizontal plane (Art. 10); and the same process will prove CD, the surface of separation

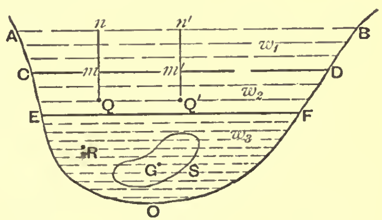


Fig. 14.

of w_1 and w_2, to be a horizontal plane. For, in the liquid w_2 take any two points, Q, Q' (as in Fig. 12, p. 26), in the same horizontal plane. Then by taking a slender horizontal cylinder having QQ' for axis, we prove that the intensity of pressure at $Q =$ that at Q'. Now taking a vertical cylinder Qmn, at Q, considering its separate equilibrium, we find that if p is the intensity of pressure at Q, and

$$Qm = x, \;\; mn = y,$$
$$p = w_2 x + w_1 y.$$

Similarly if $Q'm'n'$ is the vertical line at Q', and $Q'm' = x', m'n' = y'$,

$$p = w_2 x' + w_1 y'.$$

Hence
$$w_2 (x - x') = w_1 (y' - y) \; . \quad . \quad . \quad . \quad . \quad (1)$$

But $Qn = Q'n'$, i.e., $x + y = x' + y'$, $\therefore x - x' = y' - y$; so that unless $x - x' = 0$ and $y' - y = 0$, equation (1) will give $w_1 = w_2$, which is not the case, by hypothesis.

Hence we must have

$$Qm = Q'm', \text{ and } mn = m'n',$$

and since this holds for all points Q, Q' in the same horizontal plane, all points, $m, m', \ldots$ in the surface CD are at the same height above the same horizontal plane; therefore CD is a horizontal plane. Similarly, by taking two points in the same horizontal plane in the liquid w_3, we prove that EF is a horizontal plane.

If h_1 and h_2 are the thicknesses of the layers w_1 and w_2, and if R is a point in w_3 at a depth z below the surface, EF, of w_3, the intensity of pressure, p, at R is given by the equation

$$p = w_1 h_1 + w_2 h_2 + w_3 z, \; . \quad . \quad . \quad . \quad . \quad (2)$$

to which, if atmospheric (or other) pressure acts on the uppermost surface AB, must be added p_0, the intensity of this surface pressure, so that

$$p = p_0 + w_1 h_1 + w_2 h_2 + w_3 z. \quad . \quad . \quad . \quad . \quad (3)$$

Similarly for any number whatever of superposed layers. Each layer of liquid, in fact, acts as an atmosphere, producing an intensity of pressure on the next layer below it equal to

$$wh, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where w is the specific weight of the layer and h its thickness.

If the h's are measured in centimètres and the w's in grammes' weight per cubic centimètre, the above equations express p in grammes' weight per square cm.

The method of *regarding any layer of liquid, even when there is only one liquid in question, as an atmosphere producing an intensity of pressure given by* (4) *on the layer on which it rests, this intensity being then transmitted unaltered to all points below* (by Pascal's principle), is one which we shall frequently employ in the sequel.

From the general principle (*Statics*, vol. i, Art. 121) that, for stable equilibrium, any system of material particles acted upon by gravity only must arrange themselves into such a configuration that their centre of gravity occupies the lowest position that it can possibly occupy, it follows that in a system of superposed liquids of different densities they must arrange themselves so that the density of each liquid is greater than that of any one above it.

Again, if ABC, Fig. 15, represents a vertical section of a vessel of any shape into which are poured two different liquids, AB and BC, which do not mix, the system will settle down into a position in which the centre of gravity of the whole mass occupies the lowest position that it can occupy, and

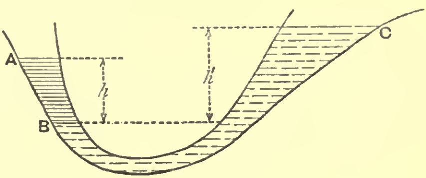


Fig. 15.

the vertical heights, h, h', of the free surfaces, A and C, above the common surface, B, of the liquids will be inversely proportional to the specific weights of the liquids.

To see the latter, we may take any point (most conveniently a point in the common surface B) and equate the intensity of pressure produced there by everything at one side of the point to the intensity of pressure produced by everything at the opposite. Thus, let w and w' be the specific weights of the liquids AB and BC, respectively; select a point, P, in the common surface B. Then if h is the difference of level between P and A, the intensity of pressure produced at P by the liquid AB and the overlying atmosphere at A is

$$wh + p_0.$$

Also, h' being the difference of level of P and C, the intensity of pressure at P produced by the right-hand liquid and the atmosphere above C is

$$w' h' + p_0.$$

There is only one intensity of pressure at P; hence these must be equal;

$$\therefore \quad w . h = w' . h', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which shows that the heights of the free surfaces above the common surface, B, of the liquids are inversely as their specific weights.

Thus, if AB is mercury and BC water, the surface C will be 13·596 times as high above B as the surface A is.

As an example, let two liquids, AB, BC, Fig. 16, be poured into a narrow circular tube held fixed in a vertical plane, the lengths of the arcs occupied by the liquids being assigned; it is required to find their positions of equilibrium.

The figure of equilibrium will be defined by the angle, θ, which the radius, OB, to the common surface of the liquids makes with the vertical, OD.

Let the angles, AOB, BOC, subtended by the liquid threads at the centre of the circle be a, a'; let their specific weights be w, w', respectively; and let r be the radius of the circle.

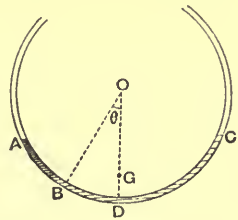


Fig. 16.

Equate the intensity of pressure produced at B by the one liquid to that produced by the other. The difference of level between B and A is

$$r\{\cos\theta - \cos(\theta + a)\},$$

and this multiplied by w is the pressure intensity at B due to the first liquid. The difference of level of B and C is

$$r\{\cos\theta - \cos(a' - \theta)\}.$$

Hence

$$w\{\cos\theta - \cos(\theta + a)\} = w'\{\cos\theta - \cos(a' - \theta)\}, \quad . \quad (6)$$

$$\therefore \quad \tan\theta = 2\frac{w'\sin^2\frac{a'}{2} - w\sin^2\frac{a}{2}}{w'\sin a' + w\sin a} \quad . \quad . \quad . \quad . \quad (7)$$

The equation (6) can be shown to express the fact that the centre of gravity of the system of two liquid threads has, in the position of equilibrium, the greatest vertical depth below O that any geometrical displacement of the two liquid threads could give it; and this is a particular case of the general property that the equilibrium position of any system of particles subject to any frictionless constraints assumes the lowest position that any arrangement of the particles can give it.

12. Pressure on a Plane Area. Let $ABCD$, Fig. 11, p. 24, represent a plane area occupying any assigned position in a heavy homogeneous liquid whose free surface is xOy.

Then if w (pounds' weight per cubic inch, suppose) is the specific weight of the liquid, and z (inches) is the depth, PN, of any point below xOy, the intensity of pressure at P, due solely to the weight of the liquid, is wz. Hence (case 5, p. 24) the resultant pressure on one side of the area is

$$A \,.\, \bar{z} \,.\, w, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\alpha)$$

where A (square inches) is the magnitude of the area, and $\bar{z}$ is the depth, GQ (inches), of its centre of area below the free surface.

If on the free surface, xOy, there is intensity of pressure (atmospheric or other) of p_0 (pounds' weight per square inch), this pressure will produce its resultant, Ap_0, at G, and the total pressure on one side of the area is

$$A\,(\bar{z}w + p_0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\beta)$$

As before pointed out (p. 25) the pressure (α) due to the liquid does not act at G, but at some point lower down.

If a plane area, S, Fig. 14, p. 29, occupies an assigned position in a liquid on the surface of which are superposed given columns of other liquids, the resultant pressure on the area is easily found. For, if $\bar{z}$ is the depth of the centre, G, of area *below the surface EF*, of the liquid w_3, the pressure of this liquid is $A\bar{z}w_3$, where A is the magnitude of the area. Also the column AD produces a resultant pressure equal to Ah_1w_1, where h_1 is the thickness of the column; the second column produces Ah_2w_2; so that the total pressure on S is

$$A\,(h_1w_1 + h_2w_2 + \bar{z}w_3); \quad . \quad . \quad . \quad . \quad . \qquad (\gamma)$$

and similarly for any number of liquids, the resultant pressure will be

$$A\,(h_1w_1 + h_2w_2 + h_3w_3 + \ldots + \bar{z}w_n), \quad . \quad . \quad . \qquad (\delta)$$

where $\bar{z}$ is the depth of G *below the surface of the liquid*, w_n, *in which the area lies.*

EXAMPLES.

1. If a plane area, occupying any position in a liquid, is lowered into the liquid by a motion of translation unaccompanied by rotation, show that the point of application of the resultant pressure on one side of the area rises towards the centre of area, G, the more the area is lowered. (See Fig. 19.)

Draw the horizontal plane CD, touching the boundary of the area at its highest point, and consider the pressures due separately to the layer between CD and the free surface, AB, and to the mass of liquid below CD. Since there is no change in the position of the area relative to the liquid below CD, this latter pressure will always act with constant magnitude and point of application, I_0; but the pressure of the superincumbent layer, always acting at G, increases in magnitude with x, the distance between AB and CD. Hence of the two parallel forces—at I_0 and G—the first remains constant, while the second continually increases; their resultant, therefore, gets nearer and nearer to G as CD is lowered.

2. Calculate in pounds' weight per square inch the intensity of pressure at a depth of 100 feet in water, neglecting atmospheric pressure.

43·4·

3. A vertical cylinder 1 foot in diameter communicates by a tube with a vertical cylinder 1 inch in diameter; and both contain water. A load of 1 ton is placed on the surface of the water in the large cylinder; what force must be applied to the surface of the water in the small one so that the water may stand 20 feet higher in the latter than in the former cylinder?

Ans. About $8\frac{3}{4}$ pounds.

4. If the load on the water in the small cylinder is removed, how much higher will the water stand in this cylinder than in the larger one?

Ans. 45·6 feet.

5. Calculate in pounds' weight per square inch the intensity of pressure at a depth of 8 inches in mercury, neglecting atmospheric pressure.

3·93·

6. The ram of a hydraulic accumulator is 1 foot in diameter, and it descends 7 feet in a minute while the water is working an engine and developing 12 horse-power; what is the mean intensity of pressure in the water?

Ans. 500 pounds' weight per square inch. (Take $\pi = \frac{22}{7}$ and 1 horse-power = 550 ft.-lb. weight per second.)

7. A triangular area of 100 square feet has its vertices at depths of 5, 10, and 18 feet below the surface of water; find the resultant pressure on one face of the area, the atmospheric intensity being 15 pounds' weight per square inch.

127·12 tons' weight.

8. Find the depth of a point in water at which the intensity of the water pressure is equal to that due to the atmosphere.

About $34\frac{1}{2}$ feet.

9. A rectangular vessel 1 foot high, one of whose faces is 6 inches broad, is filled to a height of 4 inches with mercury, the remainder being filled with water; find the total outward pressure against this face, the atmospheric intensity being 15 pounds' weight per square inch.

About $1117\frac{1}{2}$ pounds' weight.

10. Into a vessel containing mercury is poured water to a height of 8 inches above the mercury. If a rectangular area 6 inches in height is immersed vertically so that part lies in the mercury and part in the water, find the length of the area immersed in the mercury when the liquid pressure on this portion is equal to that on the portion in the water.

Nearly 1·46 inches.

11. A beaker containing liquid is placed in one pan of a balance, and is counterpoised by a mass placed in the other pan. If a solid body suspended by a string held in the hand is then immersed in the liquid, what will be the effect on the balance?

If the string sustaining the solid is attached to the arm from which the pan containing the beaker is suspended, and the system counterpoised by a mass in the other pan, will the state of the balance be the same whether the body is immersed in the beaker or not?

Ans. In the first case the pan containing the beaker will descend; in the second case there will be no change.

12. A circular tube whose plane is vertical (Fig. 16) contains a column of liquid of specific weight w; on the ends of the column rest two piston-heads of weights P and Q, fitting the tube exactly and freely movable along it; find the position of equilibrium.

Let a be the angle subtended at the centre by the thread of liquid and θ the angle made with the vertical by the radius drawn to the plug P; then

$$\tan\theta = \frac{W(1-\cos a)+aQ\sin a}{W\sin a + a(P+Q\cos a)},$$

where W is the weight of the liquid.

13. A straight glass tube of small bore is bent so that the two portions, AB, BC, are at right angles; it is held in a vertical plane with the point B downward, and the branch BC inclined at the angle a to the horizon; into AB is poured a liquid of specific weight w, the length of the column being l; into BC is poured a liquid (w', l'); find the position of equilibrium.

The length of the branch BC occupied by the liquid of specific weight w is $\dfrac{wl - w'l'\tan a}{w(1+\tan a)}$.

14. If the bent tube is rotated in a vertical plane about B, find the locus described by the point of contact of the two liquids.

Result: if P is this point and $PB = r$, $\theta =$ angle made by PB with the horizon, P being assumed to lie in the branch AB, the locus is the curve

$$r(\sin\theta + \cos\theta) = l'\cos\theta - \frac{w}{w'}l\sin\theta.$$

15. A rectangular area, $LMRS$, Fig. 17, whose plane is vertical, has one side, LM, in the free surface of water; show how to divide the area, by horizontal lines into n strips on each of which the water pressure shall be the same.

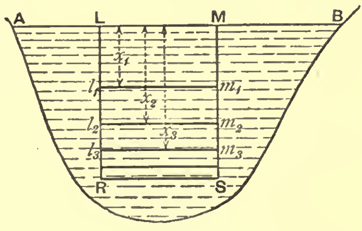


Fig. 17.

Let $LM = a$, $LR = h$,

$w =$ specific weight of liquid, and measure the depths, $x_1, x_2, x_3, \ldots$ of the successive lines of division, $l_1 m_1$, $l_2 m_2$, $l_3 m_3$, ... each from the surface.

Then the pressure on the rectangle $Lm_1 = \frac{1}{n}$ (pressure on LS);

$$\text{,, ,, ,, ,,} \quad Lm_2 = \frac{2}{n}(\quad \text{,, ,,} \quad);$$

$$\text{,, ,, ,, ,,} \quad Lm_3 = \frac{3}{n}(\quad \text{,, ,,} \quad);$$

and so on. Thus, instead of calculating the pressures on the separate strips, Lm_1, L_1m_2, L_2m_3, ... and equating them to $\frac{1}{n}$ of the pressure on LS, we take successive rectangles each having one side LM in the free surface. This is simpler.

Now the pressure on LS is $ah \cdot \frac{h}{2} \cdot w$; the pressure on Lm_1 is $ax_1 \cdot \frac{x_1}{2} \cdot w$; pressure on Lm_2 is $ax_2 \cdot \frac{x_2}{2} \cdot w$; hence

$$x_1^2 = \frac{1}{n} h^2, \quad \therefore \quad x_1 = h\sqrt{\frac{1}{n}},$$

$$x_2^2 = \frac{2}{n} h^2, \quad \therefore \quad x_2 = h\sqrt{\frac{2}{n}},$$

. . . .

$$x_r^2 = \frac{r}{n} h^2, \quad \therefore \quad x_r = h\sqrt{\frac{r}{n}}.$$

16. A triangular area, ABC, has its vertex A in the surface of water, its plane vertical, and its base BC horizontal; divide the area by horizontal lines into n strips on which the water pressures shall be equal.

Ans. The depths of the successive lines of division are

$$h\left(\frac{1}{n}\right)^{\frac{1}{3}}, \; h\left(\frac{2}{n}\right)^{\frac{1}{3}}, \; h\left(\frac{3}{n}\right)^{\frac{1}{3}}, \ldots h\left(\frac{r}{n}\right)^{\frac{1}{3}}, \ldots.$$

17. A parallelogram is immersed vertically in water with one side horizontal and at a depth of 6 feet below the surface, the opposite side being at a depth of 14 feet; find the horizontal line which divides the area into two parts equally pressed.

The line is about 4·8 feet below the upper side.

18. $ABCD$ is a parallelogram placed vertically in water with the side AB, which is 9 feet long, at a depth of 6 feet, and the side CD at a depth of 18. P is a point in AB distant 1 foot from A. Show how to draw through P a line across the area dividing the area into two parts equally pressed.

The required line cuts CD at a distance of 2 feet from C.

19. $ABCD$ is a trapezium whose parallel sides are AB and CD; the first is placed in the surface of water, and the second is below at a depth of 15 feet, the plane of the figure being vertical. $AB = 14$ feet, $CD = 16$. Find a point P in the area such that the pressure on PAB should be equal to that on PCD.

P lies anywhere on a horizontal line at a depth of 12 feet.

20. $ABCD$ is a trapezium immersed as in the last case; find a point P in the area such that the pressures on PAD and PBC shall be equal.

In this case the areas PAD and PBC must be equal; hence the locus of P is the right line joining the middle points of AB and CD. If the pressure on PAD is n times that on PBC, the locus of P is still a right line.

21. $ABCD$ is a parallelogram with the side AB in the surface of the liquid and the side CD below. Show how to draw from the corner D a right line across the area dividing the area into two parts equally pressed.

If the line cuts BC in P, we have

$$PC : BC = 3 - \sqrt{3} : 2.$$

22. In the last show how to draw the line through D so that the pressure on the triangle DPC shall be to that on the trapezium $ABPD$ as 5 to 7.

P bisects BC.

23. A trapezium whose plane is vertical has one of the parallel sides in the free surface of a liquid; divide the area by a horizontal line into two parts on which the liquid pressures are equal

Let a, b be the parallel sides, the former lying in the surface; let $h =$ height of trapezium; let $c = b - a$, and $x =$ depth of the required line; then x is given by the equation

$$2c(2x^3 - h^3) + 3ah(2x^2 - h^2) = 0. \quad . \quad . \quad . \quad . \quad (1)$$

The root of this equation which is relevant lies between $\frac{h}{2^{\frac{1}{2}}}$ and $\frac{h}{2^{\frac{1}{3}}}$, which, respectively, correspond to the cases of examples 15 and 16.

When $b = 0$, or the area is a triangle with its base in the surface and vertex down, the values of x in (1) are $\frac{h}{2}$, $\frac{h}{2}(1 \pm \sqrt{3})$, the first of which alone is relevant to the problem, since the latter two give, respectively, a value of x which is $> h$, and a negative value of x—both of which are *physically* impossible.

24. A cube is filled with a liquid, and held with a diagonal vertical; find the pressures on one of the lower and one of the upper faces.

$\frac{2}{\sqrt{3}} W$ and $\frac{1}{\sqrt{3}} W$, where $W =$ the weight of the liquid in the cube.

25. A circular area is immersed in a homogeneous liquid, a tangent to the circle lying in the free surface, A being the highest point of the circle; draw a chord, BC, of the circle perpendicular to the diameter through A so that the pressure on the triangle ABC shall be a maximum.

The distance of BC from A is $\frac{5}{6}$ of the diameter.

26. A triangle has its base, BC, in the free surface of a liquid, and its vertex A, down; find a point, O, in its area such that the pressures on BOC, COA, AOB shall be proportional to three given numbers.

If the pressures on these areas are to the pressure on ABC in the ratios $\rho_1 : \rho_2 : \rho_3$, and h is the depth of A, the point O is the intersection of a horizontal line at a depth $h\sqrt{\rho_1}$ with a line drawn from A to a point, P, in BC such that

$$\frac{BP}{PC} = \frac{\rho_3}{\rho_2}.$$

27. A semicircular area is placed in water with its diameter in the surface; show how to divide it into n sectors about the centre on each of which the pressure is the same.

Divide the diameter into n equal parts; the extremities of ordinates at the points of division determine the sectors.

28. A triangular area, ABC, occupies any position in a liquid; find a point, O, in its area such that the liquid pressures on the parts BOC, COA, and AOB shall be proportional to three given numbers.

Let a, β, γ be the depths of A, B, C below the free surface; let the ratios of the pressures on the above areas, respectively, to the pressure on the whole triangle ABC be ρ_1, ρ_2, ρ_3; let z be the depth of O, and put x for $z+a+\beta+\gamma$; then x is determined from the cubic

$$\frac{\rho_1}{x-a} + \frac{\rho_2}{x-\beta} + \frac{\rho_3}{x-\gamma} = \frac{1}{a+\beta+\gamma}. \quad . \quad . \quad . \quad (1)$$

Assuming $a > \beta > \gamma$, the value of x in this equation which is $> a$ is the only one relevant, because the values which are between a and β and between β and γ give negative values of z.

The position of O is completely defined by its areal coordinates, i.e., by the ratios of the areas BOC, COA, AOB to the area ABC. If these ratios are l, m, n, respectively, the equations are

$$l(z+\beta+\gamma) = \rho_1(a+\beta+\gamma), \quad . \quad . \quad . \quad . \quad (2)$$

and two similar, where z is $la+m\beta+n\gamma$. When z is known from (1), l is found from (2); &c.

13. Centre of Pressure. Hitherto we have been occupied with the calculation of the *magnitude* of the resultant pressure on one side of a plane area. We have now to consider the point of the area at which this resultant pressure acts. Except in the case in which the plane of the area is horizontal, this point—which is called the *centre of pressure*—is always lower in the area than G, the centroid, or 'centre of gravity', of the area.

The position of the centre of pressure on a given area varies with the position (depth, orientation, &c.) of the area in the fluid; and before determining its position in a few simple and frequently occurring cases, we shall lay down a general principle, founded on the remark near the middle of p. 31, which is often of great assistance in calculation. When a plane area—or, indeed, any surface whatever—occupies any position in a liquid, we may draw any horizontal plane whatever in the liquid and consider the column of liquid above this plane as playing the part of an atmosphere—i.e., as producing at all points below the plane a constant intensity of pressure, which is transmitted in virtue of Pascal's Principle. The most convenient horizontal plane for this purpose is one through the *highest point* of the given area.

Thus, for example, if *nrm*, Fig. 18, is any plane area whose plane is vertical in a liquid, and we wish to find the magnitude and point of action of the resultant pressure on one side of this area, we may draw a horizontal plane, CD, touching the contour of the area at its highest point, n, and then consider separately the pressures due to the layer of liquid between AB and CD and to the body of liquid below CD.

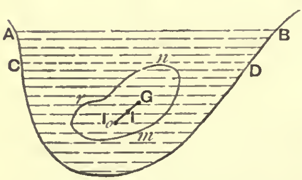


Fig. 18.

With regard to the layer *ACDB*, if x is its thickness, we know that it produces at all points on *CD* and at all points below (Art. 11) an intensity of pressure equal to

$$w \cdot x;$$

and since this pressure is uniformly distributed over the area $n r m$, its resultant is (case 1, Art. 9)

$$A w x \text{ acting at } G, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where G is the centre of area of nrm and A the magnitude of the area.

Hence, if we knew the magnitude and point, I_0, of application of the pressure of the liquid below *CD*, we should have the magnitude and point, I, of application of the pressure of the whole liquid below *AB* on the area by a simple composition of two parallel forces acting at G and I_0. This we shall presently illustrate by a few simple examples.

Thus we obtain the following construction for the centre of pressure, I, on a plane area (Fig. 19) occupying any position in a liquid: through the highest point, n, on the contour of the figure, draw a horizontal plane, *CD*, the free surface of the liquid being *AB*; from the centroid, G, (or 'centre of gravity') of the figure draw a vertical line meeting these planes in P and Q; suppose I_0 to be the (known) position of the centre of pressure if the surface of the liquid were *CD*; draw QI_0, and from P draw PI parallel to QI_0, meeting GI_0 in I. This point I is the required centre of pressure on the area. We shall presently proceed to illustrate this method by some simple examples.

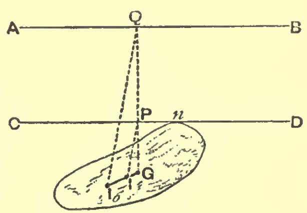


Fig. 19.

14. Special Cases of Centre of Pressure.

(1) *To find the position of the centre of pressure on a plane parallelogram, whose plane is vertical, with one side in the free surface.*

Let $ABDC$, Fig. 20, be the parallelogram. Let the area be divided into an indefinitely great number of indefinitely narrow strips, of which $mnsr$ is the type, and let E and F be the middle points of the sides AB and CD. Then the middle point of every strip lies on the line EF. Also if x is the depth of the strip ms below AB, and w the specific weight of the liquid, the intensity of pressure is the same at all points in the strip and (Art. 10) equal to wx, and the resultant pressure on the strip acts at its middle point, i.e., at the intersection, f, of ms with EF. Hence the resultant pressure on the whole parallelogram acts at some point on EF. Also, since the areas of the strips are all equal, the series of pressures on them are simply proportional to their distances from AB; therefore (case 2, p. 23) the point of application of the resultant pressure is $\frac{2}{3}$ of FE from E. Denote this point by T (see Fig. 21). Then

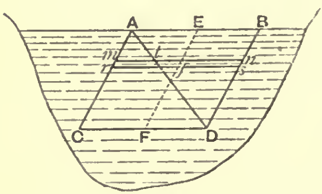


Fig. 20.

$$ET = \tfrac{2}{3} FE. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

If h is the height of the parallelogram, and $\bar{p}$ the perpendicular distance of the centre of pressure, T, from the surface

$$\bar{p} = \tfrac{2}{3} h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a')$$

If the plane of the parallelogram is not vertical, the same point, T, will still be the centre of pressure. For, if the area is inclined to the vertical at any angle, θ, and x is the perpendicular distance of f from AB, we have

$$x = fE \, . \cos\theta \, ;$$

and as θ is the same for all the strips, the pressures on them will still be proportional to their distances fE, &c.

(2) *To find the position of the centre of pressure on a plane triangle having one side in the free surface, and vertex down.*

Let ABD be the triangle. Divide the area, as before, into an indefinitely great number of strips, of which ts is the type. Let x be the perpendicular distance of this strip from the base AB. Now compare this with another strip, $t's'$, whose perpendicular distance from D is also x. Let h be the height of the triangle, $a = AB$, $k =$ the indefinitely small breadth of each strip. Then $tn = \frac{h-x}{h} \cdot a$; so that (Art. 12) the pressure on this strip is

$$\frac{ka}{h} x(h-x) w. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But this is also the pressure on the second strip, $t's'$. For, $t'n' = \frac{x}{h} a$, and the depth of $t'n'$ is $h-x$; therefore (1) is the pressure on this strip. Since each strip is pressed at its middle point, and since all the middle points lie on ED, the resultant acts at some point on ED. Also we have just seen that the pressures along ED are equal at two points such that the distance of one from $E =$ the distance of the other from D. Hence (case 4, p. 24) the resultant pressure acts at the *middle point*, M, of ED (see Fig. 21); that is,

$$EM = \tfrac{1}{2} ED. \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

If $\bar{p}$ is the perpendicular distance of M from the surface

$$\bar{p} = \tfrac{1}{2} h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta')$$

Also, whatever be the angle of inclination of the plane of the triangle to the vertical, the same point, M, is the centre of pressure.

(3) *To find the position of the centre of pressure on a plane triangle having a vertex in the free surface and its base horizontal.*

Let ACD (Fig. 21) be the triangle. Then a combination of the two results just proved will enable us to find Q, the centre of pressure. For, complete the parallelogram $ABDC$. Then the pressure on the parallelogram is the resultant of the pressures on the two triangles ACD and ADB. Let AF bisect CD and let DE bisect AB. Let T be the centre of pressure on the parallelogram, and M that of the triangle ADB. Then the force at T is the resultant of one at M and one at Q. Join the point, M, of application of one of the two parallel forces to the point, T, of application of the

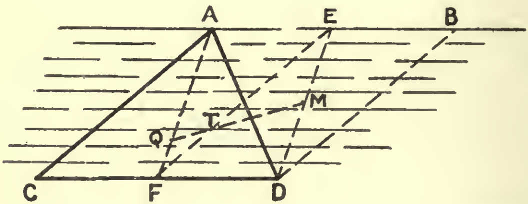


Fig. 21.

resultant, and produce MT to meet AF in Q. Then Q is the point of application of the pressure on ACD.

Now $$\frac{QF}{EM} = \frac{FT}{TE} = \tfrac{1}{2}, \quad \therefore\ QF = \tfrac{1}{2}\,EM,$$

$$\therefore\ AQ = \tfrac{3}{4}\,AF. \quad \ldots\ldots \quad (\gamma)$$

If h is the height of the triangle, and $\bar{p}$ the perpendicular distance of Q from the *free surface*,

$$\bar{p} = \tfrac{3}{4}\,h\,; \quad \ldots\ldots \quad (\gamma')$$

and, as before, the point, Q, of application of the resultant pressure is the same whatever be the inclination of the plane of the triangle to the vertical.

The result might have been deduced directly from case 3, p. 23. For, if the area be divided into strips, we have $mt = \frac{x}{h}a$, where $a = CD$, and x is the perpendicular from A on mt. Hence the pressure on the strip mt is $\frac{a}{h}wx^2$, so that the pressures along AF are proportional to the squares of the distances of their points of application from A. The resultant, therefore, acts at a point $\frac{3}{4}$ of the way down along AF.

These three simple cases, combined with the principle (see p. 31) of regarding any column of liquid as an atmosphere, producing its resultant pressure at the centre of area, will suffice for calculations concerning the centres of pressure of many plane polygonal and other figures occupying any positions in a liquid.

Thus, let the area be nrm, Fig. 18, p. 42; and suppose that, if all the liquid above the horizontal plane CD is removed, we know the depth, $\bar{p}_0$, of the centre of pressure, I_0, of the remaining liquid below CD. Then, if $\bar{z}_0$ is the depth of G below CD, A = magnitude of the area, w = specific weight of the liquid, the pressure, P_0, at I_0 is

$$A\bar{z}_0 w.$$

Let x = the thickness of the column AD. Then the pressure due to this column = Axw, and it acts at G. The resultant pressure (at I) is of course the sum of these forces; and if $\bar{p}$ is the depth of I below AB, we have, by the theorem of plane-moments,

$$\bar{p} = (\bar{p}_0 + x)\frac{\bar{z}_0}{\bar{z}_0 + x} + x, \quad . \quad . \quad . \quad . \quad . \quad (\delta)$$

the point I dividing I_0G so that

$$\frac{II_0}{IG} = \frac{x}{z_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\epsilon)$$

(4) *To find the position of the centre of pressure on a plane triangle occupying any position in a liquid.*

Let ABC, Fig. 22, be the triangle; let A be its area, and α, β, γ the depths of its vertices below the free surface of the liquid.

Take any point, P, in the area ABC, and let x, y, z be the lengths of the perpendiculars from P on the sides BC, CA, AB of the triangle. The point P is the centre of gravity of three masses proportional to the areas BPC, CPA, and APB placed, respectively, at the vertices A, B, C; for if the line AP produced meets BC in m, we have by Euclid VI. 1,

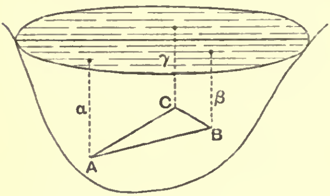


Fig. 22.

$$Bm : mC = \text{area } BAm : \text{area } mAC;$$

and also $$Bm : mC = BPm : mPC;$$

therefore $$Bm : mC = \text{area } BAP : \text{area } APC.$$

Hence the centre of gravity of the three masses above named lies somewhere on AP; similarly it lies somewhere on BP, and it is, therefore, the point P itself. If the sides BC, CA, AB are denoted by a, b, c, the masses are proportional to ax, by, cz. Now let ζ be the length of the perpendicular from P on the free surface of the liquid, and use the theorem of plane-moments (p. 19) thus:

masses	distances from free surface	products
ax	α	$a\alpha x$
by	β	$b\beta y$
cz	γ	$c\gamma z$

$$\therefore \zeta = \frac{a\alpha x + b\beta y + c\gamma z}{2A}$$

where A is the area of the triangle ABC, since the sum of the first column is $2A$.

Now the intensity of pressure at P is ζw; and we see that this is equal to the sum of the three terms

$$\frac{a\alpha}{2A}wx, \quad \frac{b\beta}{2A}wy, \quad \frac{c\gamma}{2A}wz,$$

in which the multipliers of x, y, z are the same for all points P in the area.

The term $\frac{a\alpha}{2A}wx$ would be the intensity of pressure at P if the triangle ABC were placed vertically with the side BC in the free surface of a liquid whose specific weight is $\frac{a\alpha}{2A}w$; the term $\frac{b\beta}{2A}wy$ would be the intensity of pressure at P if the triangle ABC were placed vertically with the side CA in the free surface of a liquid whose specific weight is $\frac{b\beta}{2A}w$; and the term $\frac{c\gamma}{2A}wz$ would be the intensity of pressure at P if the triangle were placed vertically with AB in the free surface of a liquid of specific weight $\frac{c\gamma}{2A}w$. Hence we can regard the actual distribution of pressure on the area ABC

(Fig. 23) as produced by a superposition of these three fictitious pressure systems.

But if BC were placed in the surface of a liquid of specific weight $\frac{a\,a}{2\,A}w$, the centre of pressure would be at i_1 (Fig. 23) which is the middle point of the bisector, AA', of

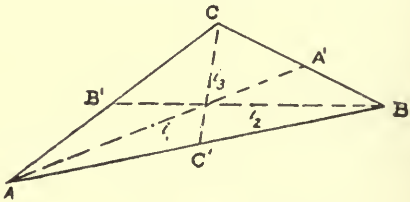


Fig. 23.

the side BC; and if p is the length of the perpendicular from A on BC, the total amount of the pressure acting at i_1 would be (Art. 12)

$$A \cdot \frac{p}{3} \cdot \frac{a\,a}{2\,A} w, \text{ or } \tfrac{1}{3} A a w.$$

Similarly, the total forces at i_2 and i_3 would be $\frac{1}{3} A \beta w$ and $\frac{1}{3} A \gamma w$. The actual pressure, then, on ABC in Fig. 23 is in magnitude and line of action the resultant of these three forces, supposed acting at i_1, i_2, and i_3. We thus get the result that *the centre of pressure coincides with the centre of gravity of three particles placed at the middle points of the bisectors of the sides, their masses being proportional to the depths of the corresponding vertices.*

This is not quite the most convenient representation. The force $\frac{1}{3} A a w$ at i_1 can be replaced by $\frac{1}{6} A a w$ at B' and $\frac{1}{6} A a w$ at C'; that at i_2 can be replaced by $\frac{1}{6} A \beta w$ at C' and $\frac{1}{6} A \beta w$ at A'; and that at i_3 by $\frac{1}{6} A \gamma w$ at A' and

$\frac{1}{6} A\gamma w$ at B'. Thus the forces are now transferred to the middle points, A', B', C', of the sides, and their magnitudes are $\frac{1}{6} A(\beta+\gamma) w$, &c.; so that if we denote the depths of A', B', C' below the free surface by ξ, η, ζ the forces at A', B', C' whose centre coincides with the centre of pressure are

$$\tfrac{1}{3} A\xi w, \quad \tfrac{1}{3} A\eta w, \quad \tfrac{1}{3} A\zeta w.$$

Hence we get one of the most useful rules in Hydrostatics, which we shall call the *Particle Rule*—

the centre of pressure on a triangular area which occupies any position in a homogeneous liquid coincides with the centre of gravity of three particles placed at the middle points of the sides, their masses being proportional to their depths below the free surface.

If a plane area consists of two or more triangles, the

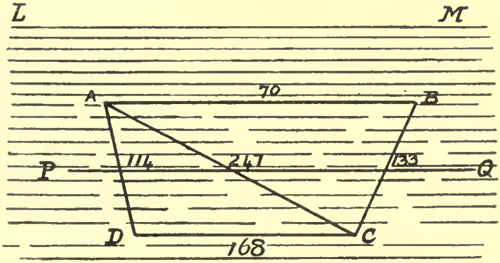


Fig. 24.

particles to be placed at the middle points of their sides are to have masses proportional to the products of the several areas and the depths of the middle points. Thus, for example, suppose the plane area to be a trapezium $ABCD$ (Fig. 24) whose parallel sides are $AB = 42$, $CD = 30$, with a perpendicular distance 18 between them, the side AB being horizontal and at a depth 10 below the free surface, LM.

Break the area into two triangles, *ABC* and *ACD*. The areas of these are proportional to 7 and 6, so that the particles at the mid points of *ABC* can be taken as having masses 70, 133, 133; those at the mid points of *ACD* are 114, 114, 168. There is a double particle equal to 133 + 114 at the middle point of *AC*.

Now since the trapezium can be broken up into infinitely narrow horizontal strips the resultant pressure on each of which acts at its middle point, the centre of pressure must lie somewhere on the line joining the middle points of *AB* and *CD*; so that we have merely to find its distance from some horizontal plane. The simplest plane to take is the horizontal plane through *PQ* which contains three of the particles. Using, then, the principle of plane-moments, and taking distances below *PQ* as positive, we have the scheme:

masses	distances from PQ	products
70	−19	−70 × 90
114	0	0
247	0	0
133	0	0
168	19	168 × 19
732	—	98 × 19

Hence the distance of the centre of pressure below *PQ* is

$$\frac{98 \times 19}{732}, \text{ or } 2{\cdot}54.$$

The centre of pressure is, then, the point of intersection of a horizontal line 2·54 below *PQ* and the line joining the mid points of *AB*, *CD*. This example serves as a type of the calculation for all plane areas that can be broken up into triangles.

The proof of the Particle Rule given above was communicated to the author by the late W. S. McCay, Fellow of Trinity College, Dublin; the rule itself had been given previously in a paper by the late Dr. E. J. Routh.

15. Pressure on Circular Area. Let a circular area, *AEBF*, be immersed vertically in a liquid with its highest point, *A* (Fig. 25), in the surface. It is required to find the centre of pressure, *I*, on this area. This point lies on the

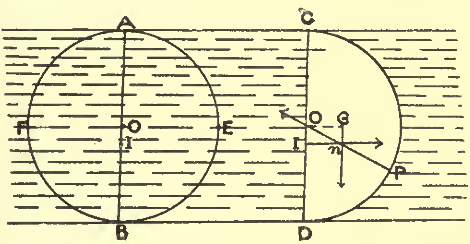


Fig. 25. Fig. 26.

vertical diameter, *AB*. Imagine a hemisphere to be constructed on the given area, and consider the separate equilibrium of the water contained within this hemisphere. The forces keeping this water in equilibrium are—(1) its weight acting vertically through its centre of gravity; (2) the pressure of the outside water acting over the plane base *AEF* and producing its resultant at *I*; and (3) the pressure of the outside water acting all over the curved surface of the hemisphere. This last produces a resultant acting through the centre *O*, since at each element of the curved surface the pressure is normal to the surface.

Let Fig. 26 represent a plane section of the hemisphere of Fig. 25 made by a plane through *AB* perpendicular to the plane of the paper. The circular base *AEBF* is represented by the line *CD*. Now the centre of gravity, *G*, of a homogeneous hemisphere of radius r is $\frac{3}{8}r$ from the centre (*Statics*, vol. i, Art. 173), and the weight of the water acting through G is $\frac{2}{3}\pi r^3 w$; also the magnitude of the pressure at I on the plane base is (Art. 12) $\pi r^3 w$; and the resultant of these two forces must act through O, because it is equal and opposite to the resultant action on the curved surface. Hence

$$\frac{\pi r^3 w}{\frac{2}{3}\pi r^3 w} = \frac{nI}{IO},$$

where n is the point of meeting of the two forces. Now $nI = OG = \frac{3}{8}r$,

$$\therefore\ IO = \tfrac{1}{4}r,$$

i.e. the centre of pressure is one-quarter of the radius below the centre. From this by the principle of Art. 13 we obtain the position of the centre of pressure when the circular area is at any depth.

16. Semicircular Area. In precisely the same way we can find the centre of pressure on a semicircular area whose

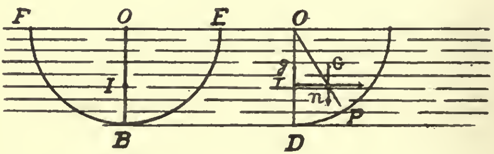


Fig. 27. Fig. 28.

bounding diameter is in the free surface. Let EF be the bounding diameter (Fig. 27), and consider the separate equilibrium of the liquid contained within the quarter-sphere having the semicircle EFB for base. Let Fig. 28 represent, as before, a section of this quarter-sphere through OB perpendicular to the plane of the paper. The centre of gravity, G, lies on the radius which is inclined at 45° to

OD, and its perpendicular distance, gO, from the base is the same as for a hemisphere, i. e.

$$gO = \frac{3}{8}r = gG.$$

The weight of liquid in the quarter-sphere is $\frac{1}{3}\pi r^3 w$, and the pressure on the base EBF is $\frac{1}{2}\pi r^2 . \frac{4r}{3\pi} \cdot w$, since (*Statics*, Art. 165) the distance of the centroid of a semicircular area from the centre is $\frac{4r}{3\pi}$. Also the liquid pressures all over the curved surface pass through O; hence

$$\frac{\frac{2}{3}r^3 w}{\frac{1}{3}\pi r^3 w} = \frac{nI}{IO} = \frac{\frac{3}{8}r}{IO},$$

$$\therefore\ IO = \frac{3\pi}{16}r.$$

EXAMPLES.

1. A triangular area whose height is 12 feet has its base horizontal and vertex uppermost in water; find the depth to which its vertex must be sunk so that the difference of level between the centre of area and the centre of pressure shall be 8 inches.

Four feet. (Use the principle in p. 31.)

2. Find the depth of the centre of pressure on a trapezium having one of the parallel sides in the surface of the liquid.

If the side a is in the surface and b below, h being the height of the trapezium, the depth of the centre of pressure is

$$\frac{h}{2}\frac{3b+a}{2b+a};$$

and it lies, of course, on the line joining the middle points of a and b.

3. In the last example find the position of the centre of pressure by geometrical construction.

(Break up the area into a parallelogram and a triangle, or two triangles.)

4. The plane of a trapezium being vertical, and its parallel sides horizontal, to what depth must the upper side be sunk in a liquid so that the centre of pressure shall be at the middle point of the area?

Ans. The parallel sides being a and b, of which the upper, a, must be the greater, the required depth $= \frac{b}{a-b}h$, where $h =$ height of the trapezium.

5. A rectangular area of height h is immersed vertically in a liquid with a side in the surface; show how to draw a horizontal line across the area so that the centres of pressure of the parts of the area above and below this line shall be equally distant from it.

The line must be drawn at a depth $(\sqrt{5}-1)\frac{h}{2}$.

6. $ABCD$ is a trapezium whose parallel sides, AB and CD, are 16 and 32 feet long, respectively, their perpendicular distance being 12 feet. AB is horizontal, at a depth of 20 feet below the free surface of water in which the area is immersed vertically, CD being below AB. Find the position of the centre of pressure.

The centre is 1·1 feet below the mid horizontal line of the figure.

7. Given that when a circular area of radius r is immersed vertically in water with its highest point in the surface the centre of pressure is $\frac{r}{4}$ below the centre, find the position of the centre of pressure on a vertical circular area of radius 2 feet, its centre being at a depth of 6 feet below the free surface.

2 inches below the centre.

8. The parallel sides of a trapezium are 40 and 30 feet long, and the height is 24 feet. If the figure is immersed vertically in water with the parallel sides horizontal and the longer uppermost, what must be the depth of this side below the free surface so that the centre of pressure shall lie on the horizontal centre line?

Ans. 72 feet.

9. What must be the depth of the upper side of this trapezium so that the centre of pressure shall be—

(*a*) 8 inches above the horizontal centre line?

(*b*) $\frac{4}{7}$ feet above this line?

Ans. The first position is impossible at any depth; the second requires an infinite depth.

10. *ABCD* is a trapezium whose parallel sides, *AB*, *CD*, are 30 and 16 inches long, respectively; the sides *BC* and *DA* are 15 and 13; and the figure is placed in water with *A* in the surface and *AB* vertical. Find the position of the centre of pressure.

Its depth is 17·54, and its distance from the vertical centre line is ·75, towards *AB*.

11. A triangular area *ACB* is placed vertically in water with the point *C* in the surface; the area being turned in its plane round *C*, prove that, so long as the area remains completely immersed, the centre of pressure describes a right line in the area, this line being parallel to *AB* at a distance equal to $\frac{1}{4}$ (height of triangle).

Apply the particle rule. If in any position x and y are the masses of particles at the mid points of *BC* and *CA*, the particle at the mid point of *AB* is $x+y$.

12. *ABC* is an isosceles triangle, $CA = CB$; it is placed vertically in water with *A* in the surface and *AB* vertical; prove that the c. p. is vertically below the centre of gravity at a distance $\frac{AB}{12}$.

Use the particle rule.

13. *ABCD* is a parallelogram with *AB* in the surface and *CD* down. Lines are drawn from *D* across to points, *P*, on *BC*. Find the locus of the centre of pressure of the triangle *DPC* as *P* varies.

14. A plane area in the form of a regular hexagon is placed vertically in water with one side in the surface; find the position of the centre of pressure by an application of the *particle rule.*

Its depth is $\frac{23}{36}$ of the height of the hexagon.

15. A rectangular vessel of height h contains liquid of specific weight w' to a height c, the remainder being filled with a liquid of specific weight w; prove that the distance of the centre of pressure on one of the vertical faces from the base is

$$\frac{1}{3}\cdot\frac{h^3w+c^3(w'-w)}{h^2w+c^2(w'-w)}.$$

16. The water at one side of a rectangular dock gate of breadth b stands at a height a, and the water at the other side at a height c; find the magnitude and line of action of the resultant water pressure on the gate.

If w is the weight of a unit volume of water, the magnitude is $\frac{1}{2}w\ (a^2-c^2)b$, and the line of action is at a height $\frac{1}{3}\ \frac{a^3-c^3}{a^2-c^2}$ above the bottom of the gate.

17. The gates of a canal lock are each $12\frac{1}{2}$ feet wide, and the breadth of the lock is 24 feet; the water at one side is 18 feet high and at the other side 12; prove that the magnitude of the thrust between the gates is about 56 tons' weight.

18. A plane area of any form is immersed vertically in water with its highest point in the free surface; and in this position h is the depth of the centre of gravity and $h+c$ that of the centre of pressure. If the area is lowered into the water without rotation and with uniform velocity v, prove that the vertical velocity of the centre of pressure at the time t, reckoned from the initial position, is

$$v\left\{1-\frac{ch}{(h+vt)^2}\right\}.$$

19. A circular area of radius r is immersed vertically in a liquid, the depth of the centre of the area below the surface of the liquid being h; show from the consideration of the separate equilibrium of the hemisphere of liquid having the given area for base that the depth of the centre of pressure on the area, below its centre, is $\frac{r^2}{4h}$.

20. A triangular area ABC of height h is immersed vertically in water with C in the surface and AB horizontal; the area is divided by a horizontal line PQ into two parts on which the

pressures are equal. Prove that the depth of the centre of pressure on the lower part is

$$\tfrac{3}{2}h\left(1-\frac{1}{2^{\frac{4}{3}}}\right).$$

21. A circular area of radius r is immersed vertically in water, its centre being at a depth h; O is the lowest point of the area. Show how to draw a horizontal chord, PQ, of the circle so that the depth of the centre of pressure on the triangle OPQ is a minimum.

Result. The depth is $\left(\frac{\sqrt{6}}{2}-1\right)(h+r)$. If, however, this is $<h-r$, that is, if $h>\frac{3+2\sqrt{6}}{5}r$, the minimum will correspond to a chord PQ of infinitesimal length at the highest point of the circle. The *maximum* depth corresponds, of course, to an infinitely short chord touching at O.

17. Self-acting Sluice. If an aperture of any shape (rectangular, circular, &c.) is made in a vertical wall or a lock-gate at one side of which there is a mass of water rising to any height, and the aperture is closed by a rigid plane surface movable about a horizontal axis in its plane, this axis can be so fixed that when the level of the water rises to any assigned height above the top of the sluice the sluice will open, let out the water, and prevent any increase in the height of the water.

It is obvious that the axis must be fixed at the centre of pressure on the sluice corresponding to the assigned level of the water.

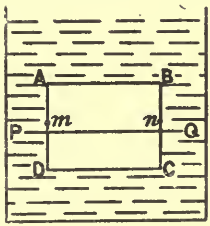


Fig. 29.

Thus if the sluice is the rectangle $ABCD$ whose sides are horizontal and vertical, and if $AD=6$ feet, while the sluice is to open when the level of the water is 6 feet above AB, we can find the depth of the centre of pressure below the line mn which bisects AD and BC by breaking the area into two triangles whose areas may be taken as 1, 1; and the depths of middle points as 2, 3, 3, 3, 4, so that the depth of c. p. below mn is $\frac{(4-2)3}{2+3+6+3+4}$, i. e. $\frac{1}{3}$ of a foot.

EXAMPLES.

1. If a rectangular sluice 3 feet high is to open when the level of the water rises $1\frac{1}{2}$ feet above its top, where must the axis of the sluice be fixed?

Ans. 3 inches below the centre of the rectangle.

2. If the height of the sluice is h, and if it is to open when the level of the water rises to a height x above its top, where must the axis be fixed?

Ans. $\frac{1}{6} \cdot \frac{h^2}{2x+h}$ below the centre.

3. Supposing a rectangular vessel whose base is horizontal to be divided into two water-tight compartments by means of a rigid diaphragm movable round a horizontal axis lying in the base of the vessel; if water is poured into the compartments to different heights, find the horizontal force which, applied to the middle point of the upper edge of the diaphragm, will keep this diaphragm vertical, and find the pressure on the axis.

Let a be the length of the axis, c the height of the vessel, h and h' the heights of the water in the compartments $(h > h')$; then the required force is $\frac{a}{6c}(h^3 - h'^3)w$, and the pressure on the axis is $\frac{1}{2}a(h^2 - h'^2)w - \frac{a}{6c}(h^3 - h'^3)w$.

18. Lines of Resistance. Supposing Fig. 30 to represent a vertical transverse section, $ABCD$, of an embankment which is pressed by water on the side AB (assumed vertical), if we take any horizontal section, PQ, of the embankment and consider the equilibrium of the portion, $QPAD$, above this section, we see that it is acted upon by its weight and also by the water pressure which is a horizontal force acting at a point two-thirds of the way down AP. Taking the resultant of these two forces, its line of action, RS, intersects the section PQ in a point R, the calculation of the position of which is of great importance in the construction of reservoirs.

As the section PQ varies in position, the point R describes a curve which is called a *line of resistance*; and it is considered essential to the stability of the wall AB that this curve must cut each horizontal section, PQ, in

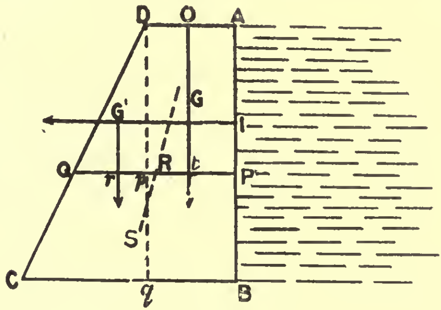


Fig. 30.

a point, R, which lies somewhere within the *middle third* of that section; that is, if we divide PQ into three equal parts, R must lie somewhere within the middle part.

Our figure represents a comparatively simple case—that in which the vertical cross-section of the embankment is a trapezium $ABCD$ with the side next the water vertical. If we make $AD = 0$, we have the case of a triangular section.

Let $AB = h$, $AD = a$, $BC = a + c$; draw Dpq vertical, and consider the area $APQD$ as composed of a rectangle, $APpD$, and a triangle pQD; let $AP = y$; let w = specific weight of water, w' = sp. weight of the masonry; let O be the mid point of AD, and take as axes of reference horizontal and vertical lines through O. Let the figure represent a portion of the embankment one unit of length thick—the thickness being measured perpendicularly to the plane of the figure. Now the water pressure against AP is $\frac{wy^2}{2}$, acting at the centre of pressure I, which is $\frac{y}{3}$ above P; the weight of $APpD$ is $w'ay$ acting through its middle point, G; the weight of pQD is $\frac{w'cy^2}{2h}$ acting through the centre of gravity G' and cutting PQ in a point r such that $pr = \frac{1}{3}pQ = \frac{cy}{3h}$.

The resultant of these forces must be counteracted by the stress exerted over the section PQ on the portion $APQD$ by the lower portion, $PBCQ$; so that the resultant of this stress must be a force acting through R and opposite to the resultant of the above three forces.

The most simple way to find the position of R is to express the fact that the algebraic sum of the moments of these three forces about R is zero. Let $tR = x$; then by moments we have

$$\frac{wy^2}{2}\cdot\frac{y}{3} + \frac{w'cy^2}{2h}\left(\frac{cy}{3h} + \frac{a}{2} - x\right) - w'ay\,.\,x = 0.$$

If we put $w' = nw$, where n will usually be between 2 and 3, this equation is

$$y\left[\left(\frac{1}{n} + \frac{c^2}{h^2}\right)y - \frac{3c}{h}x + \frac{3ac}{2h}\right] = 6ax, \quad . \quad . \quad (1)$$

and this gives the locus of R as the section PQ varies. We see that the locus is a hyperbola passing through O, having one asymptote horizontal and the other parallel to the line indicated between the brackets.

In the special case in which $AD = 0$, the locus reduces to the right line

$$\left(\frac{1}{n} + \frac{c^2}{h^2}\right) y = \frac{3c}{h} x. \quad \ldots \quad (2)$$

If $BC = AD$, that is if the vertical cross-section is a rectangle, $c = 0$, and the locus is the parabola

$$y^2 = 6an.x, \quad \ldots \ldots \quad (3)$$

whose vertex is at O.

We have assumed that the reservoir is full.

To find where the curve (1) cuts the base BC, put $y = h$, and we have

$$x = \left(\frac{h^2}{n} + \tfrac{3}{2}ac + c^2\right)/3(2a+c),$$

and for safety this must be $< \frac{1}{6}(a+4c)$.

Thus for a rectangular cross-section the condition for stability is $h < a\left(\frac{w'}{w}\right)^{\frac{1}{2}}$.

Examples of Lines of Resistance.

1. The vertical cross-section of an embankment is a rectangle, and the water reaches to a distance c from the top; show that the equation of the line of resistance referred to horizontal and vertical lines through the mid point of the top side of the rectangle is

$$(y-c)^3 = \frac{6w'}{w} axy,$$

where a is the breadth of the rectangle, w and w' being the specific weights of water and the masonry.

2. The vertical cross-section of an embankment is a rectangle of height h and breadth a; find the greatest height to which the reservoir can be filled so that the embankment shall be safe from failure by tilting over the outer edge.

Ans. $\left(\frac{3w'}{w} a^2h\right)^{\frac{1}{3}}$.

3. The vertical cross-section of an embankment is the trapezium in Fig. 24, p. 51. Given $AD = 6$ feet, $BC = 20$, $AB = 16$, mass of 1 cubic foot of masonry = 140 pounds, find the point in which the line of resistance cuts the base.

At a distance of 8·56 feet from B.

4. If in the same figure $AD = 6$, $AB = 18$, $w' = 140$ lb., find the least length of BC so that the line of resistance may cut the base within the middle third.

$BC = 10·8$.

5. If in the same figure the straight line AC is replaced by any curve, $x = \phi(y)$, show that the equation of the line of resistance is

$$2x\int_0^y \phi(y)dy = \frac{wy^3}{3w'} + \int_0^y [\phi x]^2 dy.$$

6. If the curve AC is a parabola, so is the line of resistance.

19. Stress in thin pipes. When a hollow tube, or pipe, is filled with water derived from an elevated reservoir, the very considerable water pressure in the tube produces a transverse tension in the tube which tends to split it.

Suppose that Fig. 31 represents a transverse section of the tube, and consider the separate equilibrium of a small part of the tube made by two close radial sections OP and OQ inclined at the angle POQ or θ, and contained between two close planes perpendicular to the axis of the tube—i. e. by the plane of the figure and one above it at the small height ϵ. In reality we imagine the figure to lie midway between these close planes.

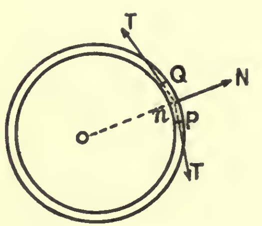


Fig. 31.

Now the normal pressure N causes the portions of the material outside P and Q to exert a tearing force, T, at P and at Q. Also N acts along On, the bisector of POQ.

Resolving along On for equilibrium, we have

$$N = 2T\sin\frac{\theta}{2}, \text{ or } N = T\theta, \text{ nearly} \quad . \quad . \quad . \quad (1)$$

Now if p is the intensity of the water pressure, $N = p \,.\, r\theta \,.\, \epsilon$, where r is the inner radius, since the area pressed is $r\theta \,.\, \epsilon$. Also if t is the intensity of the tearing force—i. e. the magnitude of this force per unit area—$T = t \,.\, \epsilon \,.\, \tau$, where τ is the (small) thickness of the tube. Hence we have from (1)

$$t\tau = pr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The bursting of the pipe depends on the magnitude of t, which is tabulated for pipes of various materials in *tons' weight per square inch* or other appropriate units.

If instead of a thin *cylinder* we have a thin *spherical* surface subject to internal pressure, p, the intensity of tearing stress is only *half* of that for the cylinder of same radius.

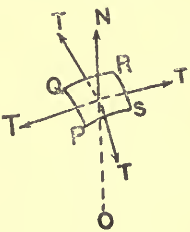


Fig. 32.

Let Fig. 32 represent a small patch of the spherical surface included by four great circles; then, considering its separate equilibrium, we shall have four T's instead of two, similarly related to the normal pressure N, so that by resolving along the normal we have now evidently

$$N = 4T\sin\frac{\theta}{2}, \text{ or } N = 2T\theta,$$

which gives

$$t \,.\, \tau = \tfrac{1}{2}pr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thus in the case of a cylindrical boiler closed at both ends by spherical caps, if the thickness is the same throughout, the tendency to burst at the side is twice as great as that at the ends.

In the ease of *thick* pipes the tension T (or 'hoop stress') varies throughout the thickness, and the preceding results do not tell us anything about the magnitude of the stress at a definite point in the substanee.

In (2) and (3) it must be understood that p means the *excess* of internal over external intensity of pressure: the external pressure will often be that of the atmosphere.

Examples.

1. Assuming that the greatest intensity of stress permissible in a metal pipe is 5,000 lb. weight per square inch, find the greatest intensity of pressure allowable inside a pipe 1 foot in diameter and $\frac{1}{6}$ inch thick.

Result. 138·8 lb. weight per square inch.

2. Water is conveyed in pipes from a height of 400 feet; the diameter of the pipes is 1 foot, and the maximum allowable stress is 2,800 lb. weight per square inch; what is the least thickness of pipe necessary?

Result. ·37 inch.

3. A thin elastic spherical envelope, of radius r, contains air at atmospheric pressure; if n times the mass of the original air is forced into it, find its new radius, assuming that the increase of surface is proportional to the intensity of tension.

Result. If r' is the new radius, and $r'^2 - r^2 = kT$, r' is found from the equation

$$2r'^2(r'^2 - r^2) = kp_0[(n+1)r^3 - r'^3].$$

CHAPTER IV

PRESSURE ON CURVED SURFACES: PRINCIPLE OF BUOYANCY

20. Principle of Buoyancy. If any curved closed surface, M (Fig. 5, p. 10), be traced out in imagination in a fluid acted upon by gravity, the pressures exerted on all the elements of this surface by the surrounding fluid have a single resultant, which is equal and opposite to the weight of the fluid enclosed by M.

This is evident, because the fluid inside M is in equilibrium under its own weight and the pressure exerted on its surface by the surrounding fluid; hence this pressure must reduce to a vertical upward force equal to the weight of the fluid inside M and acting through the centre of gravity of this fluid.

This is obviously true whatever be the nature of the fluid—liquid or gaseous, homogeneous or heterogeneous.

If the curved surface M is not one merely traced out in imagination in the fluid, but the surface of a solid body displacing fluid, the result is the same—

the resultant pressure of a heavy fluid on the surface of any solid body M is a vertical upward force equal to the weight of the fluid which could statically replace M, and this force acts through the centre of gravity of this replacing fluid.

Let Fig. 33 represent the solid body, which we may imagine to be a mass of iron, of solid rock, or any other substance, the surrounding fluid being water, air, or any fluid acted upon by gravity. The body is represented as held in its position by cords attached to fixed points, *C*, *D*, ..., and the arrows represent pressures exerted on its surface by the fluid at various points.

Now it is quite clear that if the body were replaced by any other one having exactly the same surface and occupying exactly the same position, the pressure on each element of its surface would be identically the same as before, of whatever substance the new body may be. If the new body were of wood, or instead of being solid were a thin hollow shell, it might be necessary to keep it in the position represented by means which prevent its rising up out of the fluid; but we are not at all concerned with the forces which keep the body *M* in this position; our object is merely to ascertain the resultant, if any, of the *fluid pressures* exerted in the given position on its surface.

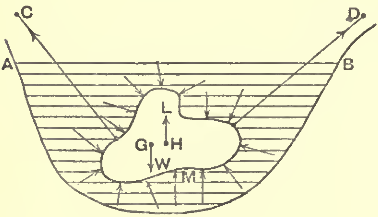


Fig. 33.

In general, a number of forces acting in various lines which do not lie in one plane have no single resultant: their simplest reduction is to *two* forces whose lines of action do not meet (*Statics*, vol. ii, chap. xiii). But it is remarkable that the pressures exerted on the various elements of any closed surface by a heavy fluid *have* a single resultant; and the truth of this we see by imagining the place occupied by *M* to be occupied by a portion of *the fluid* *itself*, placed in the vacancy without disturbing any of the surrounding fluid.

With regard to this replacing fluid, observe two things: firstly, it is in equilibrium; secondly, it is kept so by its own weight and the very same system of pressures as that which acted on the body M, since this body and the replacing fluid present identically the same surface to the surrounding fluid. Hence, then—

the system of pressures has a single resultant which is a vertical upward force equal to the weight of the statically replacing fluid and acting through the centre of gravity of this fluid.

The centre of gravity, H, of the replacing fluid is called the *centre of buoyancy*; and, so far as the general principle of buoyancy is concerned, there is no relation between H and the centre of gravity, G, of the body; nor is there any relation between the weight, W, of this body and the weight, L, of the displaced fluid.

If the fluid is water, or any homogeneous liquid, the resultant pressure is the weight of the liquid which would flow into the vacant space if M were removed; but if the density of the fluid is different in different layers, we must not imagine the replacing fluid to be that which would flow in when M is removed, but rather to be a continuation of the surrounding fluid placed in the vacancy without any disturbance of the external fluid, and having the same surfaces of equal density as this fluid. The distribution of this replacing fluid is unique and determinate, as will be subsequently proved.

Cor. 1. Pressure of uniform intensity exerted over any closed surface produces no resultant.

For, imagine the closed surface to be one traced out in a perfectly weightless fluid—or a very light gas—whose surface is subject to any pressure.

The intensity of pressure will be uniform throughout the whole fluid, and therefore over the given surface; and by what has just been said, the resultant pressure over this surface is equal to the weight of the enclosed fluid—that is to say, zero.

The result of this Corollary may also be thus stated: given any closed curve, plane or tortuous, in space; if a surface of any size and shape be described having this curve for a bounding edge, and if pressure of uniform intensity be distributed over one side of this surface, the resultant of this pressure is the same whatever the size and shape of the surface.

Hence if the given bounding curve is plane, the resultant pressure on any surface having it for a bounding edge is the same as the resultant pressure on the plane area of the curve.

COR. 2. *The principle of Archimedes.*

The particular case in which the solid body M which displaces fluid *is in equilibrium solely under the action of its own weight and the fluid pressure over its surface* furnishes the Principle of Archimedes.

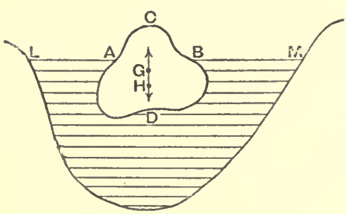


Fig. 34.

The resultant of the system of fluid pressures must then be exactly equal and opposite to the weight of the displacing body.

Thus, let Fig. 34 represent a heavy body whose centre of gravity is G, floating in equilibrium in a heavy fluid. The surface over which the fluid pressure is exerted is ADB, which is not a closed surface; but, as there is no pressure due to the fluid exerted over the free surface, *LM*, of the fluid, we can suppose the immersed surface *ADB* to be closed by the section of the body made by the horizontal plane *AB*. Hence the resultant of the pressures is the weight of the fluid that would fill the space *ADB*; and if *H* is the centre of gravity of this fluid, the resultant pressure acts up through *H*, so that *G* and *H* must be in the same vertical line. Hence there are two distinct conditions of equilibrium of a body floating freely in a heavy fluid, viz.

(1) *the weight of the body must be equal to the weight of the fluid which it displaces*; and

(2) *the centre of gravity of the body and the centre of gravity of the fluid that would statically fill its place* (*centre of buoyancy*) *must be in the same vertical line.*

The student must observe that the principle of Archimedes applies to a *closed* surface, which is completely surrounded by liquid—or, as pointed out above, to an unclosed surface, such as that of the immersed portion of a ship, which may be supposed to be closed by a surface of zero pressure. It does not apply to such a case as the following:

ABC (Fig. 35) is a solid body with a perfectly flat base, *BC*, resting on the base of a vessel into which water is poured, the fit of the bases being so accurate that no water flows in between them. The water pressure on the body, so far from an *upward* force, is a *downward* one, the reason being that the surface pressed is not closed and subject all over to water pressure, and that it cannot be assumed to be closed by a surface of zero pressure, as the intensity of the pressure at the level of *BC* is not zero. We might as well expect the elevated curved portion of the bottom of a champagne bottle to

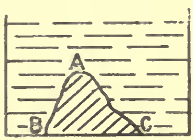


Fig. 35.

be raised by the liquid as expect the body in Fig. 35 to be urged upwards. It is important also to bear in mind that the principle of Archimedes is not identical with the *general* principle of buoyancy, but only an application of the latter to the particular case of a body floating freely in a liquid under the influence of no forces but its own weight and the pressure of the surrounding liquid.

We have hitherto supposed that the only fluid displaced by the body is that represented in the vessel below the surface LM; but if above this there is air, whose weight is considered, there is also displaced a volume of air represented by ACB, and the resultant effect of the air is to produce an *upward* vertical force, even though (as in the figure) the air pressure exerted by the air actually in contact with the displacing body should be a *downward* force; for, we must remember that the surface LM of the lower fluid is all subject to air pressure which is (by Pascal's Principle) transmitted undiminished all through this fluid, so that the lower part, ADB, of the surface of the body is really acted upon all over by air pressure of constant intensity. Now by Cor. 1, the resultant of this system of air pressures on the curved surface ADB is the same as if the pressure was applied over the lower side of the plane area AB in which the surface LM cuts the body. The resultant air pressure is, therefore, an *upward* force equal to the weight of the air that would statically fill the space ACB, and it acts through the centre of gravity of this air.

The case of a balloon floating in the air is also an instance of the principle of Archimedes; the force of buoyancy is the weight of the air that could statically replace all the solid portions of the balloon and the gas which it contains. It must not be supposed that, since the balloon is a comparatively small body, the intensity of the air pressure is constant all over its surface—a not unnatural error; for, if this air pressure were of constant intensity all over the surface, its resultant would be absolutely zero, as we have already seen, and there would be no force of buoyancy. If the medium surrounding a body is ever so slightly acted upon by gravitation, its intensity of pressure cannot be constant, and hence the densities of the air at the top and at the bottom of the balloon are not the same.

MISLEADING STATEMENT OF THE PRINCIPLE OF ARCHIMEDES.

A desire for conciseness of expression leads very often to misleading and erroneous statements of scientific facts. Thus, a very common reply to the question 'what is the principle of Archimedes?' is simply this—'the weight of a floating body is equal to the weight of the fluid displaced.' To show the misleading nature of this reply, take the following example:

A cylinder contains a small quantity of water whose level is AB (Fig. 36); a cylindrical block of wood, $mnpq$, whose diameter is nearly equal to that of the vessel is lowered into the water and allowed to float, if it can do so. Suppose that it does float without touching the bottom of the vessel, and occupies the position $m'n'p'q'$. There will remain a thin layer of water at the bottom between the vessel and the wood, and the remainder of the water will be forced up the sides and attain the level $A'B'$.

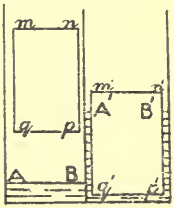


Fig. 36.

Now the weight of the floating block may be very many times as great as that of the water actually displaced—and, in fact, greater than that of all the water present. The force of buoyancy is not at all equal to the *weight of the water displaced,* but to the very much greater weight of the volume, $A'B'p'q'$, of water that would occupy the place of the immersed portion of the body without disturbing the surrounding water.

As a numerical example, suppose that the radius of the vessel is 3 cm., the radius of the cylindrical body *mnpq* 2·9 cm., the mass of this body 240 grammes, and the height of AB above the base 1 cm. In this case we find that the body will float with a layer of water about ·41 cm. below it, and that the water will rise to a height of about 9·5 cm. above the base of the vessel.

Here the fluid actually displaced is only a column of radius 2·9 and height equal to the difference of level of AB and $q'p'$, that is about ·6 cm.; and the weight of the water displaced is the weight of about 15·85 grammes, whereas the force of buoyancy is the weight of the volume $A'B'p'q'$, and is, of course, the weight of 240 grammes.

21. Introduction of Fictitious Forces. In the case in which a body is partially immersed in a fluid, or a part of the body is in one fluid and the remainder in another, it is often very convenient to introduce fictitious forces of buoyancy in one part of the calculation and to take them away in another.

Thus, suppose Fig. 34 to represent a body of which the portion ADB is immersed in water while the portion ACB is in vacuo. Then the actual force of buoyancy is due to the volume ADB of water; but we can complete the volume of the displaced water by supposing the portion ACB to be also surrounded by water, and then supposing that there is a *downward* force, in addition, due to the action of this portion ACB of water taken negatively. Thus the actual force of buoyancy—viz. an upward force at H equal to the weight of the volume ADB of water—can be replaced by an *upward* force equal to the weight of the whole volume $ADBC$ of water acting at the centre of gravity of the homogeneously filled volume $ADBC$ (not G, the c. g. of the body, unless the body is itself a homogeneous solid), together with a *downward* force equal to the weight of the volume ACB of water acting at the centre of gravity of the homogeneously filled volume ACB.

In the same way, if the portion ACB is in a liquid of specific weight w_1, and ADB in one of specific weight w_2, we may regard the force of buoyancy as consisting of an upward force equal to the weight of the whole volume $ADBC$ of the liquid w_2 together with a downward force equal to the weight of a fictitious liquid of specific weight $w_2 - w_1$ acting at the centre of gravity of the homogeneously filled volume ACB.

Examples.

1. A solid homogeneous right cone floats in a given homogeneous liquid; find the position of equilibrium, firstly, when the vertex is down and base up; and, secondly, when the base is down and the vertex up.

Let w', V, h be the specific weight, volume, and height of the cone; let w be the specific weight of the liquid, and x the length of the axis immersed when the vertex is down. Then since the volumes of similar solids are proportional to the cubes of their corresponding linear dimensions, the volume of the displaced liquid $= \frac{x^3}{h^3} V$. Hence, equating the force of buoyancy to the weight of the cone,

$$\frac{x^3}{h^3} Vw = Vw',$$

$$\therefore\ x = h\left(\frac{w'}{w}\right)^{\frac{1}{3}}.$$

In the second case, if x is the length of the axis above the liquid, the volume of the displaced liquid $= \left(1 - \frac{x^3}{h^3}\right) V$, and we have

$$x = h\left(1 - \frac{w'}{w}\right)^{\frac{1}{3}}.$$

2. A solid homogeneous isosceles triangular prism floats in a given homogeneous liquid; find the position of equilibrium in each of the two previous cases.

If x is the depth of its edge below the surface, h the height of the isosceles triangle which is the section of the prism by a plane perpendicular to the edge, and A the area of this section, since the areas of similar figures are as the squares of their corresponding linear dimensions, $\frac{x^2}{h^2}A$ is the area of the face of the immersed prism in the first case, and if $l =$ length of edge, the volume of the prism is lA, so that the volume of the immersed prism is $\frac{x^2}{h^2}V$. Hence

$$\frac{x^2}{h^2}Vw = Vw',$$

$$\therefore \quad x = h\left(\frac{w'}{w}\right)^{\frac{1}{2}}.$$

In the second case,

$$\left(1 - \frac{x^2}{h^2}\right)Vw = Vw',$$

$$\therefore \quad x = h\left(1 - \frac{w'}{w}\right)^{\frac{1}{2}}.$$

3. A uniform rod, AB, of small normal section and weight W has a mass of metal of small volume and weight $\frac{1}{n}W$ attached to one extremity, B; find the condition that the rod shall float at all inclinations in a given homogeneous liquid.

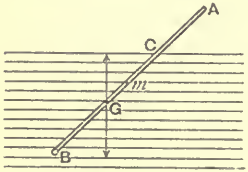


Fig. 37.

Let $AB = 2a$, let m be the middle point of AB, Fig. 37, G the centre of gravity of the rod and the metal, w' the specific weight of the rod, w that of the liquid, and s the area of the normal section of the rod.

Then $W = 2asw'$, and $BG = \frac{n}{n+1}a$. Also G must be the centre of buoyancy if the rod floats in the oblique position represented, and the length, BC, of the displaced column of liquid $= \frac{2n}{n+1}a$, so that if the weight of this column $= \left(1 + \frac{1}{n}\right) W$, both conditions of equilibrium will be satisfied, whatever be the inclination of the rod. Equating the weight of the body to the force of buoyancy,

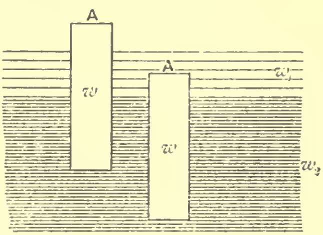
$$\left(1 + \frac{1}{n}\right) 2\,asw' = \frac{2n}{n+1}\,asw;$$

$$\therefore \quad (n+1)^2 w' = n^2 w,$$

which is the relation required between the specific weights.

4. A solid homogeneous cylinder floats, with its axis vertical, partly in a homogeneous liquid of specific weight w_1 and partly in one of specific weight w_2, the former resting on the latter; find the position of equilibrium.

Let h be the height of the cylinder, A the area of its base, w its specific weight, and c the thickness of the upper liquid column.



Figs. 38, 39.

Then if we assume the top, A, of the cylinder to project a distance x above the upper surface of the upper liquid, as in Fig. 38, and equate the weight of the cylinder to the sum of the forces of buoyancy due to the displacements of the liquids, we have

$$hw = cw_1 + (h-c-x)w_2, \quad \ldots \ldots \quad (1)$$

$$\therefore \quad x = h\left(1 - \frac{w}{w_2}\right) - c\left(1 - \frac{w_1}{w_2}\right). \quad \ldots \quad (2)$$

If $c\left(1 - \frac{w_1}{w_2}\right) > h\left(1 - \frac{w}{w_2}\right)$, we must write

$$x = -\left\{c\left(1 - \frac{w_1}{w_2}\right) - h\left(1 - \frac{w}{w_2}\right)\right\}, \quad \ldots \quad (3)$$

and it would appear that the position of equilibrium is one in which, as in Fig. 39, A is below the upper surface of the upper liquid by the distance

$$c\left(1-\frac{w_1}{w_2}\right)-h\left(1-\frac{w}{w_2}\right), \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in virtue of the usual interpretation of a negative co-ordinate in algebra.

To take a numerical case, suppose $c=\frac{1}{2}h$, and

$$w : w_1 : w_2 = 5 : 2 : 6;$$

then $x=-\frac{1}{6}h$, and it would appear that A is $\frac{1}{6}h$ below the upper surface of the upper fluid.

Now if we had originally assumed A to be, as in Fig. 39, at an unknown distance, x, below the surface, our equation would have been

$$hw=(c-x)w_1+(h-c+x)w_2, \quad . \quad . \quad . \quad . \quad (5)$$

$$\therefore \quad x=c-h\,\frac{w_2-w}{w_2-w_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which disagrees with (4), and which in the particular numerical case gives $x=\frac{1}{4}h$, instead of $x=\frac{1}{6}h$, which we had been led to expect by interpretation of the negative value (3).

Why the disagreement? Because the continuity of the values of variables in algebra and algebraic geometry finds no corresponding characteristic in the hydrostatical conditions. In fact, the supposition that the negative value (3) harmonizes with the physical assumptions leading to the first solution is untrue; for, in this solution we assume that, whatever be the unknown position of equilibrium of the body, the *whole column of the upper liquid is operative in producing its force of buoyancy*, as is evident from the first term, cw_1, at the right-hand side of (1); whereas the supposition that A is below the upper surface of this liquid is an explicit assumption that the whole column of the liquid may *not* be so operative. Hence we ought not to expect the two solutions to agree.

In the case, therefore, in which the value of x in (2) is negative, the correct result is (6) and not (3).

5. A heavy uniform bar, AB, of small cross-section is freely movable round a horizontal axis fixed at one extremity, A, at a given height above the surface of a homogeneous liquid in which the rod partly rests; find the position of equilibrium and the pressure on the axis.

Let $AB = 2a$; let h be the height of A above the liquid; let $s =$ area of cross-section of the rod; let w' and w be the specific weights of the rod and the liquid; and let $\theta =$ the angle between AB and the vertical.

Then if BC is the part immersed, the centre of buoyancy, H, is the middle point of BC. If $W=$ weight of rod, $W = 2asw'$; also

$$BC = 2a - h\sec\theta,$$

$\therefore$ the force, L, of buoyancy

$$= (2a - h\sec\theta)\,sw.$$

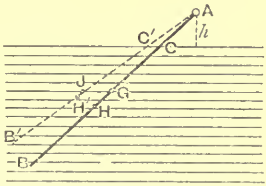


Fig. 40.

The rod is in equilibrium under the action of L, W, and the pressure at A, which last must be vertical and $= W - L$.

Taking moments about A for equilibrium,

$$W.AG\sin\theta = L.AH\sin\theta, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and if we reject the factor $\sin\theta$, i.e. omit the consideration that $\sin\theta = 0$ gives one position of equilibrium (the vertical one), we have

$$4a^2w' = (4a^2 - h^2\sec^2\theta)\,w, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore \quad \cos\theta = \frac{h}{2a}\left(\frac{w}{w-w'}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The oblique position requires w to be greater than w' and also

$$w > \frac{4a^2}{4a^2 - h^2}\cdot w';$$

so that, for example, if the bar were of metal and the liquid water, the only position of equilibrium would be the vertical one.

6. A uniform pole, AB, the linear dimensions of whose cross-section are small compared with its length, is supported by a cord attached to A and floats, partly immersed, in water; find its position of equilibrium.

Result. If $2x$ is the length of the immersed portion, w and w' the specific weights of the liquid and pole,

$$x = a\left(1 - \sqrt{1 - \frac{w'}{w}}\right).$$

Hence the length of the immersed part is independent of the inclination; but if $w' > w$, the pole must float vertically. The suspending cord always assumes a vertical position.

7. If the pole is 12 feet long, and its specific gravity $\frac{5}{9}$, and if the end A is fixed by a horizontal axis at a height exceeding 8 feet above the water, the only position of equilibrium is a vertical one. If the height of A is 6 feet, prove that the inclination of the pole to the vertical is $\cos^{-1} \cdot 75$.

8. Two uniform straight rods, of lengths $2a$, $2b$, and specific gravities s, s', respectively, are joined together to form a single straight rod; find the ratio $a : b$ so that there may be an inclined position of equilibrium when the system floats freely in water.

Result. The ratio is given by the equation

$$s(1-s)a^2 + 2s'(1-s)ab + s'(1-s')b^2 = 0,$$

or the same with $2s(1-s')$ for the coefficient of ab.

9. The specific gravity of ice is ·918 and that of sea-water 1·026; prove that the volume of the submerged portion of an iceberg is $8\frac{1}{2}$ times that of the portion above water.

10. A hollow closed cone of metal whose specific gravity is 8 is to be made of such uniform thickness that it will float in all positions wholly submerged in water; show that the thickness must be $\cdot 0109 \times h$, where h is the height of the cone.

11. If the specific weight of the metal is w', and the cone floats in all positions, wholly submerged, in a liquid of specific weight w, show that the semi-vertical angle of the cone must be $\sin^{-1}\frac{1}{3}$, and that the thickness must be

$$\frac{h}{4}\left\{1 - \left(1 - \frac{w}{w'}\right)^{\frac{1}{3}}\right\}.$$

12. A solid homogeneous cone is floating in water with its axis vertical and vertex downwards; to cause it to sink until $\frac{3}{4}$ of its axis is immersed requires a load of 50 grammes on its base; and to cause $\frac{4}{5}$ to be immersed requires 96 grammes; show that the sp. gr. of the cone is ·324 nearly.

13. A cylindrical block of wood the area of whose section perpendicular to its axis is 50 square inches floats, with the

axis vertical, in water in a vessel whose horizontal cross-section is uniform and equal to 80 square inches; the specific gravity of the wood is $\frac{2}{3}$ and the height of the block is 2 feet; how much does the level of the water rise above the position which it occupies when the block is withdrawn, and how much of the axis is under water?

Ans. 10 inches; 16 inches.

14. A block of wood of height h, cross-section B, and specific weight w' floats in a liquid of specific weight w contained in a vessel of cross-section A; how much does the level of the liquid rise after the immersion of the block, and how much of the block is immersed?

Ans. $\frac{B}{A} \cdot \frac{w'}{w} \cdot h$; $\frac{w'}{w} \cdot h$.

15. A body of weight W is allowed to float in a tank of cross-section A containing a liquid of specific weight w; how much does the level of the liquid rise?

Ans. $\frac{W}{Aw}$.

16. A block of wood, of sp. gr. $\frac{4}{9}$, in the shape of a prism whose section perpendicular to its edge is an isosceles triangle whose base is 8 inches long and height 10 inches, the length of the edge being 27 inches, is placed with its edge submerged in a tank 1 foot broad and 3 feet long; how high does the level of the water rise, and what is the depth to which the edge of the block is submerged?

Ans. $1\frac{1}{9}$ inches; $6\frac{2}{3}$ inches.

17. The section of an isosceles prism of wood perpendicular to its parallel edges is a trapezium whose parallel sides are 8 and 6 feet long, the perpendicular distance between them being 4 feet; the sp. gr. of the wood is $\frac{13}{28}$, and the prism is floating in water with the longer of the parallel sides uppermost; find the position of equilibrium.

The thickness of the immersed part is 2 feet.

[Whatever the distance between the parallel sides may be, if the other data remain unaltered, half the thickness will be immersed.]

18. $ABCD$ is a uniform rectangular board of specific weight w'; it is to float in a liquid of specific weight w with the diagonal AC submerged and horizontal when a particle of negligible volume whose mass is n times that of the board is attached at the corner B; find the condition and position of equilibrium.

If the side AD cuts the surface of the liquid at P, and if we put $DP = \frac{1}{m} . DA$, m is determined by the equation $3m^3\left(1 - \frac{w'}{w}\right) - 3m + 1 = 0$, while n and m must satisfy the equation $n = \frac{w}{w'}\left(1 - \frac{1}{2m^2}\right) - 1$.

19. A solid homogeneous hemisphere is movable round a horizontal axis which is a tangent to its rim and rests partially immersed in a liquid; find its position of equilibrium.

Result. Let h be the height of the axis above the liquid, a the radius and w' the specific weight of the solid, w being that of the liquid; then the plane of the base is inclined to the horizon at an angle θ given by the equation

$$(a - h + a\sin\theta)^2(2a + h - a\sin\theta)w = 2a^3(1 - \tfrac{3}{8}\tan\theta)w'.$$

20. A solid homogeneous body of any shape is movable round a fixed horizontal axis and rests partially immersed in a liquid contained in a trough, the section of floatation being a curve AB marked on the surface of the body; liquid is poured into the trough, the body rising by rotation round the axis which becomes immersed at increasing depths; it is observed that in the course of revolution the same curve AB becomes again the curve of floatation. Show that the specific gravity of the body must be one-half of that of the liquid.

22. Resultant Pressure on an unclosed curved surface. Suppose $BCDA$, Fig. 41, to represent any unclosed surface in a heavy fluid, and suppose its bounding edge to be a plane curve so that the surface can be closed by a plane base, represented by AB. It is required to find the resultant of the fluid pressures exerted on one side of the unclosed surface.

Closing the surface by means of the plane base AB, the resultant of the pressures all over the outside of the completely closed surface is the vertical upward force, L, represented by the line HL drawn through the centre of gravity, H, of the fluid which would fill the volume. But if P is the resultant fluid pressure on the plane base AB, acting at the centre of pressure, I, the force L is the resultant of P and the resultant pressure over the unclosed part. This latter force, Q, is therefore found by producing the lines of action of L and P to meet—at O, suppose—and drawing On and Om to represent L and P, respectively; then the required force Q is represented by the line OQ which is equal and parallel to mn.

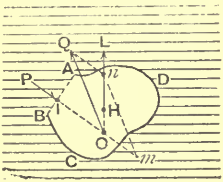


Fig. 41.

If the fluid is a homogeneous liquid of specific weight w, if A is the area of the plane base AB, $\bar{z}$ the depth of the centre of area of AB below the free surface, and V is the volume of the closed surface,

$$P = A\bar{z}w, \text{ and } L = Vw.$$

Hence if θ is the inclination of the plane base AB to the horizon,

$$Q = w\sqrt{V^2 + 2VA\bar{z}\cos\theta + A^2\bar{z}^2};$$

$$\text{horizontal component of } Q = A\bar{z}w\sin\theta,$$

$$\text{vertical component of } Q = (A\bar{z}\cos\theta - V)w.$$

EXAMPLES.

1. Suppose a right cone whose axis is vertical and vertex downwards to be filled with a liquid; find the resultant pressure on one-half of the curved surface determined by any plane containing the axis.

Let ACB, Fig. 42, be the vertical plane of section, and $ACDB$ the half of the curved surface on which we desire to find the resultant liquid pressure.

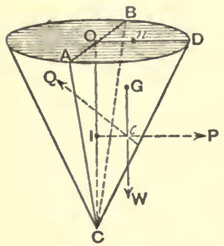


Fig. 42.

Consider the separate equilibrium of the fluid contained between this curved surface and the triangle ACB. It is kept at rest by its weight, the pressure of the remaining fluid on the area ACB acting at I, the centre of pressure on this triangle, and by the pressure of the curved surface $ACDB$. The weight acts through G, the centre of gravity of the semi-cone; and if on the diameter, OD, which is perpendicular to AB, we take the point n such that $On = \frac{4r}{3\pi}$, where r is the radius of the base, this point n is the centre of area of the semicircle ADB, so that G lies on nC and $Gn = \frac{1}{4} Cn$ (*Statics*, vol. i, Art. 163). The point I is half-way down OC (Art. 14). If P is the pressure on the triangle ACB, h = height of cone,

$$P = \tfrac{1}{3} rh^2 w; \text{ and } W = \tfrac{1}{6}\pi r^2 hw,$$

where W = weight of liquid. The lines of action of P and W meet in a point c, whose position is thus completely known; and by drawing cP and cW to represent P and W on any scale, the diagonal through c of the rectangle thus determined will represent Q, the resultant pressure of the curved surface on the fluid in the semi-cone. The line cQ is drawn to represent this pressure, and this force reversed is the pressure of the fluid on the surface.

2. If the cone is closed by a base, and the axis is held horizontal, find the resultant pressure on the lower half of the curved surface.

Result. If X and Y are the horizontal and vertical components of the resultant pressure,

$$X = \left(\frac{\pi}{2} + \frac{2}{3}\right) r^3 w,$$

$$Y = \left(1 + \frac{\pi}{6}\right) r^2 h w,$$

and the line of action of the pressure passes through a point whose distances from the base and the axis of the cone are

$$\frac{h}{4} \cdot \frac{\pi + 8}{\pi + 6} \text{ and } \frac{r}{4} \cdot \frac{3\pi + 16}{3\pi + 4}.$$

3. If a hollow cylinder is filled with liquid and held with its axis vertical, determine the magnitude and line of action of the resultant pressure on one half of the curved surface cut off by a vertical plane through the axis.

It is a horizontal force equal to $r^2 h w$ acting in a line $\frac{h}{3}$ from the base.

4. If the cylinder is closed at both ends and held with its axis horizontal, find the resultant pressure on the lower half of the curved surface.

A vertical force $= \left(2 + \frac{\pi}{2}\right) r^2 h w$.

5. In example 2 find the magnitude and line of action of the resultant pressure on the upper half of the curved surface.

If X and Y are the horizontal and vertical components of the pressure,

$$X = \left(\frac{\pi}{2} - \frac{2}{3}\right) r^3 w,$$

$$Y = \left(1 - \frac{\pi}{6}\right) r^2 h w,$$

and the line of action of the resultant passes through a point whose distances from the base and the axis of the cone are

$$\frac{h}{4} \cdot \frac{8 - \pi}{6 - \pi} \text{ and } \frac{r}{4} \cdot \frac{16 - 3\pi}{3\pi - 4}.$$

6. A spherical shell is filled with liquid; find the magnitude and line of action of the resultant pressure on the curved surface of either hemisphere cut off by any vertical central plane.

The line of action passes through the centre of the sphere; the horizontal component is $\pi r^3 w$, and the vertical $\frac{2}{3}\pi r^3 w$.

7. A spherical shell is filled with liquid; find the magnitude and line of action of the resultant pressure on each of the hemispheres into which the sphere is divided by any diametral plane.

If θ is the inclination of the plane section to the horizon, the pressure on one hemisphere is the resultant of two forces $\pi r^3 w$ and $\frac{2}{3}\pi r^3 w$, respectively perpendicular to the plane section and vertical, the lines of action of these forces including an angle θ, while the pressure on the other curved surface is the resultant of the same forces including an angle $\pi - \theta$; and both pass through the centre.

8. If a hole is made in the top of the shell and fitted with a funnel, find the height to which the funnel must be filled with the liquid in order that the resultant pressure on one of the hemispheres shall be a horizontal force.

The height $= r\left(\frac{2}{3}\sec\theta - 1\right)$.

When the unclosed curved surface which is exposed to water pressure cannot be closed by a plane base—i. e. when its bounding edge is not a plane curve—the total vertical component of pressure on one side of the surface is easily found. Thus, in Fig. 43, suppose *AB* to represent an unclosed surface, and consider the pressures (represented by the arrows) exerted on one side by the water. At the various points of the bounding edge draw vertical lines terminated by the free surface, *PQ*, of the water, and consider the separate equilibrium of the cylindrical column, *PABQ*, of the liquid. This column is kept in equilibrium by (1) its weight, (2) the horizontal pressures exerted all round its vertical surface, and (3) the pressures on its under side which are represented by the arrows in the figure *reversed* in direction.

Resolving forces vertically, we see that the vertical component of these latter forces is equal to the weight of the column standing on AB—commonly called *the weight of the superincumbent fluid.*

The curved surface may be such that some of the pressures on one and the same side of it have a *downward* and others an *upward* component, as in Fig. 44. In this case a portion of the above cylinder will be formed by lines, such as BQ,

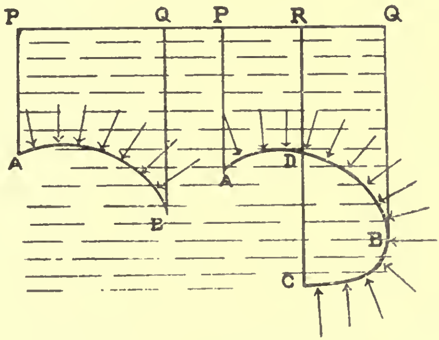


Fig. 43. Fig. 44.

which are tangents to the given surface ABC; and on the corresponding part of the surface, AB, the pressures have a *downward* vertical component equal to the weight of the column ABQ; while the pressures on the remainder, BC, have an *upward* component equal (by the same reasoning) to the weight of the column $RCBQ$. The resultant upward component is equal to

$$\text{weight of } DBC - \text{weight of } PADR$$

If in Fig. 44 the point *C* either coincides with *A* or is vertically under *A*—i. e. if the surface *ABC* is either a perfectly closed surface or one which can be closed by a vertical plane base—the column *PADR* vanishes, and the upward thrust is equal to the weight of the liquid enclosed by the surface *ABC*. Thus we are brought back to the principle of buoyancy, Art. 20.

In both Fig. 43 and Fig. 44 the pressures have also a horizontal component, but this cannot be obtained in the same simple manner; for, the term *vertical* is perfectly definite, while the term *horizontal* is not so: *vertical* means parallel to a *line*, whereas *horizontal* means merely parallel to a *plane*—only *one* vertical line can be drawn through a given point, but an *infinite number* of horizontal lines can be drawn through it.

To find the total horizontal component of the pressures on one side of an unclosed curved surface along a given horizontal line, project the bounding edge of the surface on a vertical plane perpendicular to the given horizontal direction, by horizontal lines drawn through the points of the bounding edge; then find the magnitude of the pressure exerted on the (plane) area thus obtained. This pressure, calculated by Art. 12, is the required horizontal component.

23. Moulds. The vertical thrust of a fluid is well illustrated in the case of a mould for producing a metal casting.

Suppose that it is desired to make a thin hollow cone of metal. Take a solid cone, *ABC*, of clay, plaster of Paris, or other substance, fastened to a horizontal base; and over it place a mould, *PQRS*, containing a hollow, similar, and slightly larger conical cavity, *EFGH*, with a narrow vertical aperture, *DEH*, the mould being fastened in position by weights or bolts. If the molten metal is poured in at *D*, it will fill the space between the cone and the mould, and solidify into a thin cone of metal. Now, while the metal is in the liquid state, the mould will be acted upon by pressure all over the conical shoulder *FEHG*, and the resultant vertical upward thrust of the liquid is equal to the weight of the column of the liquid standing on *FEHG* with vertical sides reaching up to the level, *PQ*, of the free surface. This column is cylindrical in shape, and is represented in section by *mFEHGn*, deducting the volume of the aperture *DEH*.

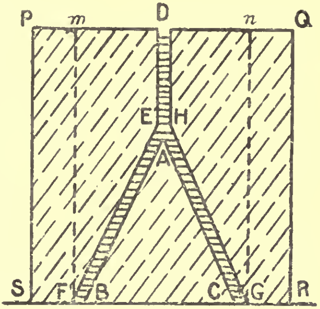


Fig. 45.

If H is the height, QR, of the mould, h the height of the cone $FEHG$, r the radius of the base of this cone, and w the specific weight of the metal, the vertical upward thrust tending to lift the mould off its base is, approximately

$$\pi r^2 (H - \tfrac{1}{3} h) w.$$

In the same way a hemispherical bowl may be cast if the solid BAC is replaced by a hemispherical body, and the space $FEHG$ is also hemispherical.

EXAMPLES.

1. A hemispherical bowl of metal 2 feet in radius is cast with a mould whose height in 3 feet; the mass of the metal per cubic foot is 480 pounds; prove that the lifting force on the mould is about $4\frac{1}{2}$ tons' weight.

2. Two hollow cones, of the same vertical angle, are joined together at their vertices so that their axes are in the same

vertical line, forming a figure like an hour-glass (Fig. 46). The vessel is closed at one end with a plane base and rests on a horizontal plane with the axis vertical. Find the height to which it must be filled with water so that the total *vertical* water thrust on the curved surface shall be zero.

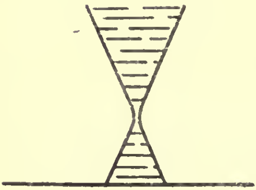


Fig. 46.

Result. If h is the height of the lower cone and H the height of the upper cone of water,

$$H^3 - 3Hh^2 - 2h^3 = 0,$$

which gives $H = 2h.$

3. A hollow frustum of a cone, 6 feet high, is placed on the ground with its base (8 feet in diameter) cemented, water-tight, to the ground; the upper extremity of the frustum has a diameter of 2 feet and is connected with a vertical tube 10 feet high which is rigidly joined all round to the frustum. If the compound vessel is filled to the top with water, what is the force tending to tear the vessel from the ground?

Ans. The weight of 204π cubic feet of water, i. e. 17·88 tons' weight.

4. A thin spherical shell of radius r and negligible weight has a small hole into which a tube is fitted; the tube is placed vertically and filled with water to a height h above the centre of the sphere, the system being kept in equilibrium in any way. Show that the whole force with which the upper hemisphere of the shell tends to separate from the lower is

$$\tfrac{1}{4} W\left(3\frac{h}{r} - 2\right).$$

A very striking illustration of the principle of buoyancy is furnished by the following experiment. Take a straight glass tube; insert into it at A (Fig. 47) another glass tube, Av, at right angles; bend the first tube at m and n at opposite sides of Av, making two constrictions at m and n, the branches of the tube being mB and nC; through the open ends B, C insert a ball c of cork soaked in paraffin and a solid marble, g, of glass or metal; close the ends B and C

of the branches. If the system is held with Av vertical the cork and the marble will fall to m and n and be stopped by the constrictions. Now pour in water through v, filling both branches. The cork will rise to B and the marble will remain at n. Let a clamp be fixed at A to the tube, and attach this clamp to a whirling apparatus so that the whole can be rotated rapidly about the vertical line Av. As the rotation increases, the cork will come down from B and try to reach A, while the marble will ascend towards C.

Let ω be the angular velocity of the tube, and suppose the cork to be at c; let r be the distance of c from the vertical axis of rotation, and let W_1 be the weight of the cork.

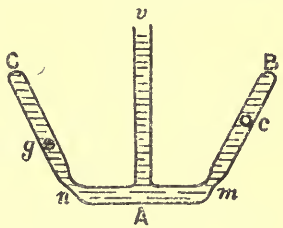


Fig. 47.

Suppose the cork to be tied to the point B by a thread, whose tension is T, while the system is rotating about Av with angular velocity ω; then the forces acting on the cork are T, W_1, N (the normal reaction of the tube) and the pressure of the surrounding liquid. This last is exactly the same as it would be on a sphere of the liquid itself which might replace the cork. Let H and V be the horizontal and vertical components of this pressure, and let W be the weight of the liquid that would occupy the place of the cork. Now since the sphere of liquid goes round in a horizontal circle without the aid of a thread but with the aid of a pressure N' from the tube (unless the sphere is so small that it is not in contact with the tube), its equation of horizontal motion is

$$W\frac{\omega^2 r}{g} = H - N' \sin a,$$

$$\therefore\ H = W\frac{\omega^2 r}{g} + N' \sin a\,; \quad . \quad . \quad . \quad . \quad (1)$$

and also for vertical equilibrium,

$$V = W + N' \cos a. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The equation of horizontal motion of the cork is

$$W_1 \frac{\omega^2 r}{g} = H - N \sin a - T \cos a,$$

$$= W \frac{\omega^2 r}{g} - (N - N') \sin a - T \cos a,$$

by (1); while vertical equilibrium gives

$$V = W_1 + N \cos a - T \sin a$$

$$= W + N' \cos a.$$

Hence we have

$$(W - W_1) \frac{\omega^2 r}{g} = (N - N') \sin a + T \cos a, \quad . \quad . \quad (3)$$

$$W - W_1 = (N - N') \cos a - T \sin a. \quad . \quad . \quad (4)$$

Eliminating $N - N'$, we have

$$T = (W - W_1) \left(\frac{\omega^2 r}{g} \cos a - \sin a \right). \quad . \quad . \quad . \quad (5)$$

This equation shows that T is negative so long as

$$\omega^2 < \frac{g}{r} \tan a,$$

and therefore the cork would require a *push* towards m to keep it in position; but when $\omega^2 = \frac{g}{r} \tan a$, no force along the tube is required; and when ω^2 increases beyond this value, T becomes positive—that is, the cork tries to go down the tube.

The case of the marble is similarly discussed; let W_2 be the weight of the marble, then for the tension of the constraining cord we have

$$T = -(W_2 - W) \left(\frac{\omega^2 r}{g} \cos a - \sin a \right)$$

$$= (W_2 - W) \left(\sin a - \frac{\omega^2 r}{g} \cos a \right), \quad . \quad . \quad . \quad (6)$$

which shows that when ω increases beyond the above-named value, T becomes negative—that is, a push downwards along Cn is required, or, in other words, the marble ascends the tube.

24. Work of Immersion. When a body displaces liquid, a certain amount of work is done against the force of buoyancy. Suppose, for example, that a solid cylinder is allowed to sink into water until it finds its position of equilibrium. Then the principle of work and energy shows that the work done on the cylinder by gravity is equal to the work which the cylinder does against buoyancy from the first to the second position.

Suppose Fig. 48 to represent a body of any form displacing a liquid contained in any given vessel. Let AB be the level of the liquid before the body was immersed, and let PQ be the level to which the liquid rises when the body is immersed. [We are not assuming that the body is *floating in equilibrium*: we assume merely that by some means it is now in the position represented.] Now it is quite clear that the volume LCN of the liquid is taken up and spread into the layer whose section in the plane of the figure is represented by $LAPR$ and $NBQS$; so that the work done against buoyancy is the work of raising the volume LCN into the position of the layer. This work is the same as that done by raising the volume $RLCNS$ into the position of the volume $PRSQBNLA$, since by this process the volume $RSNL$ would be undisturbed.

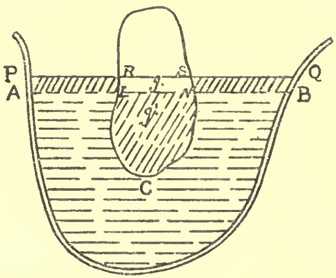


Fig. 48.

Let g' be the centre of gravity of the volume $RLCNS$, supposed filled with the liquid, and let g be the centre of gravity of the layer $PABQ$; then the work done against buoyancy is the weight of the layer $PABQ$ multiplied by the difference of level of the points g and g'.

For clearness, let us take a numerical case.

To find the work which must be done to submerge completely a cylinder of weight W, height h, and specific gravity $\frac{2}{3}$ in a cylindrical vessel of water, the area of the cross-section of this vessel being twice that of the body.

Let Fig. (1) represent the body just touching the surface of the water; Fig. (2) the body in its equilibrium position

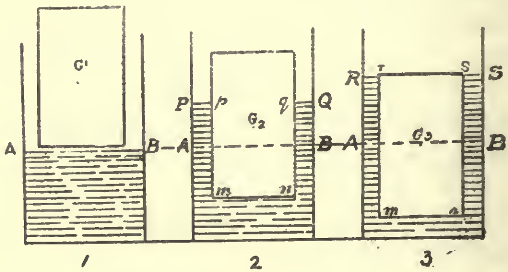


Fig. 49.

when left to itself; Fig. (3) the body completely submerged by means of a downward pressure.

Since the force of buoyancy in Fig. (2) is the weight of the volume $pmnq$ of liquid, or of the layer $PAQB$, i. e. of the volume $S \times AP$, where $S =$ area of cross-section of vessel, we have

$$S \times AP \times w = \tfrac{1}{2} S \times h \times \tfrac{2}{3} w,$$

$$\therefore\ AP = \tfrac{1}{3} h, \text{ and } pm = \tfrac{2}{3} h.$$

Now from position (1) to position (2) the work against buoyancy is the work of raising the volume $pmnq$ into the position $PABQ$; and as the centre of gravity of the first of these volumes is on the plane AB, and the centre of gravity of the second is $\frac{h}{6}$ above this plane, the weight of the volume *pmnq* being equal to W, the required work against buoyancy from (1) to (2) is $\frac{1}{6} Wh$. Denote this work by B_{12}; also denote the work against buoyancy from (1) to (3) by B_{13}. Now the volumes *rmns* and $RABS$ are equal; $\therefore AR = \frac{1}{2} h$, and the centre of gravity, G_3, of *rmns* is on the plane AB; also the weight of this volume of liquid $= \frac{3}{2} W$; hence

$$B_{13} = \tfrac{3}{2} W . \frac{h}{4} = \tfrac{3}{8} W . h.$$

If B_{23} denotes the work against buoyancy from (2) to (3), we have $B_{23} = B_{13} - B_{12}$;

$$\therefore B_{23} = (\tfrac{3}{8} - \tfrac{1}{6}) W . h$$
$$= \tfrac{5}{24} W . h.$$

This, however, is more than the work which we should have to do to sink the body from (2) to (3), because we are assisted by the work done on the body by gravity from (2) to (3). This work is $W \times$ diff. of level of G_2 and G_3; that is $\frac{1}{6} Wh$; hence, deducting this from B_{23}, we have

$$\tfrac{1}{24} W . h$$

for the work which, in addition to that contributed by gravity, is required to sink the cylinder.

As a further exercise on this same question, take this problem: *the cylinder being held by the hand in the position* (1) *and allowed to drop into the liquid, find how far it will sink before coming for an instant to rest.*

Applying the principle of work and energy to the body, since the kinetic energy in the first position $= 0$, and also that in the second position $= 0$, the work done by gravity on the body must be numerically equal to the work done by buoyancy on it. Suppose the body to sink until $AP = x$;

then, since vol. $pmnq$ = vol. $PABQ$, we see that $pm = 2x$, and the work against buoyancy from (1) to (2) is $\frac{1}{2} S \cdot 2x \cdot w \cdot \frac{x}{2}$, since the centre of gravity of $pmnq$ is on the plane AB; this work is $\frac{1}{2} Sw \cdot x^2$. To get the work done on the body by gravity, observe that, measuring all distances from the fixed plane AB, the centre of gravity, G, of the body is, in the first position, at a height of $\frac{1}{2} h$ above AB, and in the second at a height $\frac{1}{2} h - x$ above AB; and, the weight of the body being $\frac{1}{3} Shw$, the work done by gravity from (1) to (2) is $\frac{1}{3} Shw (\frac{1}{2} h - \frac{1}{2} h + x)$, or $\frac{1}{3} Shxw$. Equating this to the work against buoyancy, we have $x = \frac{2}{3} h$, a result which shows that the cylinder disappears below the liquid; but the result cannot be relied upon as expressing the final position, since we have assumed in the work that p is not above the top of the cylinder. (See a similar case, p. 77.) We must make a fresh calculation assuming that the top of the cylinder disappears to a distance z above the plane AB. The work of gravity will be $W(h-z)$, and the work against buoyancy $\frac{1}{2} Shw(\frac{3}{4} h - z)$, or $\frac{3}{2} W (\frac{3}{4} h - z)$. Equating these we have

$$z = \frac{1}{4} h,$$

in other words, three-fourths of the cylinder goes below the original level of the liquid.

If the vessel, or tank, in which the body floats is of unlimited cross-sectional area—a lake or a river—as represented in Fig. 50, the work of buoyancy is calculated very simply. Thus, let the body be immersed so that $pqmn$ is under the liquid; then the work done against buoyancy in reaching this position is simply the work of raising the volume $pqmn$ of liquid and spreading it out along the unlimited surface AB of the liquid as an infinitely thin horizontal layer. If $pm = x$ and A = area of cross-section of the cylinder, the volume raised is Ax, and the height through which its centre of gravity is raised is $\frac{1}{2}x$; so that the work done against buoyancy is $\frac{1}{2}Ax^2w$; and the work done against buoyancy in submerging it to the position in which $pm = x'$ is $\frac{1}{2}Ax'^2w$; therefore the work done between these two positions is $\frac{1}{2}A(x'^2 - x^2)w$.

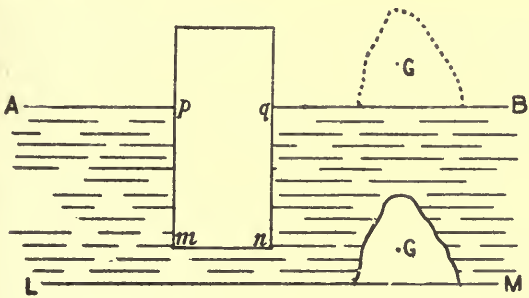


Fig. 50.

Let the first position be the freely floating one, and the second the position of complete submergence; then if w' is the specific weight of the cylinder, $x = h\dfrac{w'}{w}$, and $x' = h$; therefore the work against buoyancy from the one to the other is

$$\tfrac{1}{2}Ah^2\left(1 - \frac{w'^2}{w^2}\right)w.$$

The weight of the body itself has assisted in the submergence and done the amount $Ahw'(x' - x)$, or

$$Ah^2w'\left(1 - \frac{w'}{w}\right),$$

so that the work required to be done by an external agent to submerge the body from the floating position is

$$\tfrac{1}{2}Ah^2\left(1 - \frac{w'^2}{w^2}\right)w - Ah^2w'\left(1 - \frac{w'}{w}\right),$$

i. e. $$\tfrac{1}{2}Ah^2\frac{(w-w')^2}{w}, \text{ or } \tfrac{1}{2}Wh\frac{w}{w'}\left(1-\frac{w'}{w}\right),$$

where W is the weight of the cylinder.

EXAMPLES.

1. A solid homogeneous cylinder of height h, cross-section B, and specific weight w' floats with its axis vertical in a liquid of specific weight w contained in a tank of cross-section A; find the work required to submerge it.

Result. $$\frac{B(A-B)}{2A}h^2\frac{(w-w')^2}{w}.$$

2. A solid body of weight W, specific gravity s, and given figure rests on the bottom of a river of depth h; find the work necessary to raise it just clear of the water.

Let G be the centre of gravity of the body, Fig. 50, and let the height of G above the base be k; then the work of buoyancy, which assists in the raising is that done by the falling of a superficial layer of water into the position of the body. The volume of this layer = vol. of body $= \dfrac{W}{w'}$, where w' = sp. weight of body, ∴ the weight of the layer $= \dfrac{W}{s}$, and the original and final heights of the centre of gravity of the layer above the bottom, LM, are h and k; therefore the work of buoyancy $= \dfrac{W}{s}(h-k)$; the centre of gravity, G, of the body is raised through h; ∴ the required work is

$$Wh - \frac{W}{s}(h-k).$$

3. Calculate the work required to submerge a homogeneous sphere of specific weight $\frac{1}{2}w$ in a tank of given cross-section containing a liquid of specific weight w.

If V = vol. of sphere, r = radius, A = area of section of tank, the work is

$$Vw\left(\tfrac{5}{16}r - \frac{V}{8A}\right).$$

[Taking the three positions of Fig. 49,

$$AP = \frac{V}{2A}; \; AR = \frac{V}{A};$$

and we have

$$B_{12} = \tfrac{1}{2} Vw \left(\tfrac{3}{8} r - \frac{V}{4A}\right),$$

$$B_{13} = \;\; Vw \left(r - \frac{V}{2A}\right),$$

$$\therefore \; B_{23} = \;\; Vw \left(\tfrac{13}{16} r - \frac{3V}{8A}\right).$$

Deduct the work done on the sphere by gravity from (2) to (3), which is $\frac{1}{2} Vw \left(r - \frac{V}{2A}\right)$, and we get the result.]

4. If this sphere is held in the position (1) and allowed to drop into the liquid, how far will it descend before being brought for an instant to rest?

Ans. It will just reach the position (3) of complete submergence.

5. A solid homogeneous cone of weight W, height h, and specific gravity $\frac{8}{27}$ floats in water of indefinite horizontal extent; find the amount of work required to submerge it.

$\frac{11}{32} W.h$. [Work against buoyancy $= \frac{65}{96} W.h$; work of gravity $= \frac{1}{3} W.h$.]

6. A solid homogeneous cone of weight W, height h, and specific weight w' floats, vertex down, in an indefinite mass of liquid of specific weight w; find the work required to submerge it.

Result. $\frac{1}{4} W.h \left\{ \frac{w}{w'} + 3\left(\frac{w'}{w}\right)^{\frac{1}{3}} - 4 \right\}$.

$$\left[\text{Work against buoyancy} = \tfrac{1}{4} W.h \left\{ \frac{w}{w'} - \left(\frac{w'}{w}\right)^{\frac{1}{3}} \right\}; \right.$$

$$\left. \text{work of gravity} = W.h \left\{ 1 - \left(\frac{w'}{w}\right) \right\}. \right]$$

7. If this cone is held with its axis vertical and its vertex just touching the liquid, and is allowed to drop into the liquid,

how much of it will be submerged in the position of temporary rest?

Ans. Four times the volume submerged when the cone is floating freely in equilibrium—provided that this volume does not exceed the whole volume of the cone. If, however, $w' > \frac{1}{4}w$, let the base of the cone disappear to a depth x below the surface; then the work done against buoyancy

$$= \tfrac{1}{4} Vwh + Vwx;$$

and the work done by gravity on the cone

$$= Vw'(h+x);$$

equating these, we have

$$x = \frac{w' - \frac{1}{4}w}{w - w'} \cdot h.$$

8. A solid sphere of radius r and specific gravity s lies on the bottom of a cylindrical vessel of radius a and height h which is filled to the top with water; prove that the work required to raise the sphere just clear of the water is

$$W\left[h - \frac{4r^3}{3a^2} - \frac{1}{s}\left(h - r - \frac{2r^3}{3a^2}\right)\right].$$

(When the sphere is just clear, the height, x, of the water is $h - \dfrac{4r^3}{3a^2}$. Now take the mass-moment of the system in the first and in the second position with respect to the base. For the first consider the sphere to consist of a body of sp. gr. equal to 1 and a sphere of sp. gr. equal to $s-1$. Hence if w = weight, per unit volume, of water, the first mass-moment is $\pi w\left[\frac{1}{2}a^2h^2 + \frac{4}{3}(s-1)r^4\right]$, and the mass-moment in the second is $\pi w\left[\frac{1}{2}a^2x^2 + \frac{4}{3}s(x+r)r^3\right]$. The excess of the second over the first is the work required.)

9. A solid cone of volume V, height h, and specific weight w' floats with its vertex down in a liquid of specific weight w contained in a tank of uniform horizontal section A; find the amount of work necessary to submerge the cone.

$$\textit{Result.}\quad \frac{3Vhw'}{4}\left\{\frac{1}{3}\left(\frac{w}{w'} - 1\right) + \left(\frac{w'}{w}\right)^{\frac{1}{3}} - 1\right\} - \frac{V^2(w-w')^2}{2Aw}.$$

(Take mass-moments about the surface of the liquid in the freely floating position.)

10. A solid homogeneous body of weight W, volume V, and specific gravity s rests on the bottom of a tank of uniform cross-section A, the height of the centre of gravity of the body above the base being z; the tank contains water to a height h when the body is immersed. Prove that the work required to raise the body clear of the water is

$$W\left[\left(1-\frac{1}{s}\right)h-\left(1-\frac{1}{2s}\right)\frac{V}{A}+\frac{z}{s}\right].$$

CHAPTER V

GASES

25. Definition of a Perfect Gas. A gas may be described roughly as a fluid which can be easily compressed. A more precise mathematical definition will be given subsequently; for the present, we shall define a perfect gas as a fluid which obeys the law of Boyle and Mariotte. This law is as follows: *the temperature remaining constant, the volume of a given mass of gas varies inversely as its intensity of pressure.*

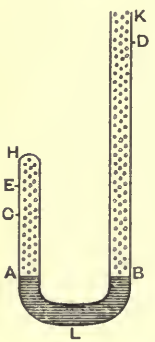


Fig. 51.

This law is proved experimentally as follows. Let *HABK* (Fig. 51) be a bent glass tube of uniform section—at least in the leg *AH* which is closed at the top. Let the gas to be experimented upon be enclosed in the branch *AH* by means of a column, *ALB*, of mercury, the branch *LBK* of the tube being open to the atmosphere. Suppose matters arranged so that when the gas in *AH* is in equilibrium of temperature with the surrounding air, after the pouring in of the mercury has ceased, the surfaces *A* and *B* of the mercury are at the same level in both branches. Then the intensity of pressure at any point in the surface *A* is equal to that at any point in *B*; so that if p_0 is the atmospheric intensity of pressure, p_0 is also the intensity of pressure of the gas in *AH*.

For simplicity denote p_0 by the height of the barometer at the time of the experiment.

Let this height be h (inches or millimetres), and let v_0 (cubic inches or cubic millimetres) be the volume of the gas AH. If w is the weight of a unit volume (cubic inch or cubic millimetre) of the mercury, we have $p_0 = w \,.\, h$.

Let us now, by pouring mercury slowly into the open branch at K, reduce the volume of the gas in AH to half its value. If $CH = \frac{1}{2} AH$, the mercury is to be poured in until its level in the closed branch stands at C after all disturbance and heating effect due to the pouring in of the mercury have subsided. If we now read the difference of level between C and the surface of the mercury in the branch LK, we shall find it exactly equal to h, the height of the barometer. Equating the intensity of pressure at C due to the imprisoned gas to the intensity of pressure due to the mercury and the atmosphere, we see that the former must be equal to $p_0 + wh$, i. e. the new intensity of pressure $= 2p_0$, while the new volume is $\frac{1}{2} v_0$.

Again, let $EH = \frac{1}{4} AH$, and let us pour mercury in at K until the volume of the imprisoned gas is EH, i. e. $\frac{1}{4} v_0$. We shall then find that the difference of level between E and the surface, F, of the mercury (not represented in the figure) in the open branch is 3 times the height of the barometer, i. e. $3h$, so that the intensity of pressure of the gas in EH is $p_0 + 3wh$, or $4p_0$.

Hence we have the following succession of volumes and pressure intensities for the gas, its temperature being the same all through,

$$(v_0, p_0), \quad (\tfrac{1}{2} v_0, 2p_0), \quad (\tfrac{1}{4} v_0, 4p_0).$$

If, in the same way, the volume is reduced to $\frac{1}{n} v_0$, the difference of level of the mercury in the two branches is found to be $(n-1)h$, so that the new intensity of pressure is np_0; and from these results we see that in each case the volume of the gas is inversely proportional to its intensity of pressure, as stated in the law.

The law of Boyle and Mariotte may also be verified in the following simple manner by means of a single straight tube, about 2 mm. in diameter.

Let AD be a tube of uniform section closed at the end A and open at D; let a portion, AB, of the tube be filled with air or other gas, and let a thread of mercury, BC, of length l, separate this gas from the external air.

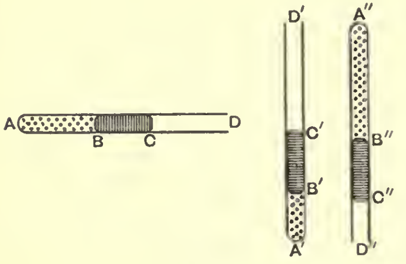


Fig. 52.

When the tube is held horizontal and all disturbance has subsided, let the volume, v_0, of the gas AB be read; its intensity of pressure is the same as that at C, i. e. p_0, the atmospheric intensity. Now let the tube be held in a vertical position with the closed end A' downwards and let the gas occupy the volume $A'B'$, or v'. Its intensity of pressure is now equal to that at B' due to everything above B', i. e. $p_0 + wl$, where w = weight of unit volume of mercury. If h is the height of the barometer during the experiment, $p_0 = wh$, and if p' is the intensity of pressure in $A'B'$,

$$p' = w(h+l).$$

Finally, let the tube be held vertically with the closed end A'' uppermost, and let the volume of the gas be $A''B''$, or v''. If its intensity of pressure is p'', the intensity at C'' is $p'' + wl$ due to everything above C''; but p_0 is also the intensity of pressure at C'' since that is a point in the external air. Hence $p'' = w(h-l)$.

Hence, as regards volume and intensity of pressure, we have the succession of states

$$\{v_0, wh\},\ \{v', w(h+l)\},\ \{v'', w(h-l)\},$$

and we find on trial that

$$v_0 h = v'(h+l) = v''(h-l),$$

according to the requirements of Boyle's law.

This law is not accurately obeyed by any known gas, but the approximation is very close in the case of all gases when they are not near the state in which, either by increase of pressure or by diminution of temperature, liquefaction begins. When any gas is near the state of liquefaction, its volume decreases more rapidly with increased pressure than it would if it followed Boyle's law. 'When it is actually at the point of condensation, the slightest increase of pressure condenses the whole of it into a liquid.' (Clerk Maxwell's *Theory of Heat*, chap. i.)

26. Graphic Representation. We can graphically represent its various states as expressed in the fundamental equation (β), thus: draw any two rectangular axes, Ov, Op, and let the volumes assumed by the gas be measured, on any scale, along Ov, while the intensities of pressure are measured on any scale along OP.

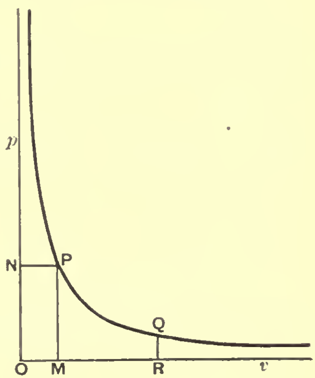


Fig. 53.

If, on these scales, OM and ON represent respectively any volume and the corresponding intensity of pressure, the point, P, whose co-ordinates are OM and ON will graphically represent the state of the gas; and all points, such as P, whose co-ordinates satisfy (β) will be found on a rectangular hyperbola passing through P and having the axes Ov and Op for asymptotes.

Thus, then, *the curve of transformation of a given mass of gas at constant temperature is a rectangular hyperbola.* Such transformation is called an *isothermal transformation.*

The figure exhibits the fact that when the intensity of pressure is infinitely increased the volume of the gas becomes infinitely small, and that when the intensity of pressure is infinitely reduced, the volume becomes infinitely great.

The first result would be strictly true for a substance whose transformations strictly follow the law for all values of p; but it will be readily understood that there exists no gas for which Boyle's law holds indefinitely, and that when the intensity of pressure is greatly increased the gas may approximate to, and actually become, a liquid.

27. Law of Dalton and Gay-Lussac. The volume of a given mass of gas may be altered by heat as well as by pressure. The law relating to this change was discovered independently by Dalton in 1801 and by Gay-Lussac in 1802; and, apparently, it was discovered fifteen years previously by M. Charles, although not published by him. It is this—

The intensity of pressure being constant, the volume of a given mass of gas, when its temperature is raised from the freezing to the boiling point of water, increases by a fraction of the volume at the first temperature, which fraction is the same for all gases.

In short, the law is that all gases have the same coefficient of expansion, and that this is independent of the magnitude of the (constant) intensity of pressure under which they expand.

The fraction in question is, with certain reservations to be mentioned presently,

$$\cdot 3665,$$

so that if we measure degrees of heat by the Centigrade thermometer, this is the fractional increase of volume for 100°. The rate of expansion per degree is also found to be uniform, and is ·003665, which we shall use in the form

$$\frac{1}{273}.*$$

Hence if v_0 denotes the volume of a given mass of any gas at 0° C., and v its volume at $t°$ C., we have

$$v = v_0\left(1 + \frac{t}{273}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whatever be the intensity of pressure (supposed constant); and if v' is its volume at t', we have

$$\frac{v}{273 + t} = \frac{v'}{273 + t'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the point from which the temperature is reckoned on the Centigrade thermometer is removed 273° below the ordinary zero, i. e. the point at which water freezes when its surface intensity of pressure is that due to a standard atmosphere (indicated by a mercurial column 760 mm. in height), the expression $273 + t$ indicates the newly measured temperature, and is always denoted by T, and called the *absolute temperature* of the substance, the new point of reckoning being called the *absolute zero* of temperature.

If it were possible to have $T = 0$, that is $t = -273°$, for the gas—supposing the substance to remain a gas at

* The fraction is more accurately $\frac{1}{272 \cdot 85}$, but the above is usually taken for simplicity. Clerk Maxwell (*Theory of Heat*) gives various values; thus, $\frac{1}{273\frac{1}{3}}$ and $\frac{1}{273 \cdot 7}$, the latter deduced from experiments of Thomson and Joule.

all temperatures with constant coefficient of expansion, $\frac{1}{273}$—equation (1) would give

$$v = 0,$$

i. e. the gas would be reduced to zero volume. As the substance does not satisfy the above supposition, but alters its state in the process of lowering the temperature, the consequence is not realized, and it would thus appear that the notion of an *absolute zero* of temperature at $-273°$ C. is a gratuitous error. Indeed, if the conception of *absolute temperature* rested on no other foundation, we might similarly argue from the coefficient of expansion of platinum, for instance, that since for this body $v = v_0\left(1 + \frac{t}{37699}\right)$, nearly, where v_0 is its volume at zero and v its volume at $t°$, if we make $t = -37699$ we shall arrive at the absolute zero of temperature.

The truth is that the measure of absolute temperature rests on quite another basis, that it is intimately connected with the coefficient of expansion of a perfect gas, and that $273 + t$ is properly to be regarded as measuring the absolute temperature of a body whose temperature indicated by a Centigrade thermometer is t.

Adopting absolute temperature, then, equation (2) gives

$$\frac{v}{T} = \frac{v'}{T'}. \qquad (3)$$

Of course in the expression of the law of Dalton and Gay-Lussac it is not necessary to signalize the particular temperature corresponding to the *freezing of water* as possessing any special reference to the expansion of gases.

The law may be stated thus: *all gases expand, per degree, by the same fraction of their volumes at any common temperature.* This is obvious because their volumes at *any* temperature, τ, will all be the same multiple of their volumes at o°, and a constant fraction of the latter will give a constant fraction of the former.

In symbols, for any gas let u be the volume at τ, v that at t, v_0 that at zero, and a the coefficient of expansion with reference to the volume at zero; then

$$v = v_0(1 + at);\ u = v_0(1 + a\tau);$$

and therefore $$v = u\frac{1 + at}{1 + a\tau} = u\frac{1 + a\tau + a(t-\tau)}{1 + a\tau}$$

$$= u\left\{1 + \frac{a}{1 + a\tau}(t-\tau)\right\}$$

$$= u\{1 + \beta(t-\tau)\},$$

where $\beta = \dfrac{a}{1 + a\tau}$, so that β is obviously the rate of expansion of the gas reckoned as a fraction of the volume u; and if a is the same for all gases, so is β.

It is remarkable that a is the same for all gases when far removed from their condensing points, i. e. from the liquid states, and that it is independent of the intensity of pressure under which the expansion takes place.

Clerk Maxwell (*Theory of Heat*) points out that *if the law of Dalton and Gay-Lussac is true for any one intensity of pressure, and if the law of Boyle holds, it follows that the former law holds for all intensities of pressure.*

This is very easily proved thus. Let v_0 be the volume of a given mass of gas at $(0°, p)$, i. e. p is its intensity of pressure, and let the law of Dalton hold for this pressure intensity; then if v is its volume at (t, p)

$$v = v_0(1 + at).$$

Now, keeping t constant, alter p to p'; then by Boyle's law the new volume, u, is given by the equation

$$u = v_0(1 + at)\cdot\frac{p}{p'}\cdot$$

But if v_0 at $(0°, p)$ were altered by keeping its temperature zero and changing its intensity of pressure to p', its value, u_0, would be $v_0 \cdot \frac{p}{p'}$ by Boyle's law; so that the last equation gives

$$u = u_0(1 + at),$$

and therefore Dalton's law holds for p' if it holds for p.

With regard to the *accuracy* of the law of Dalton and Gay-Lussac, M. Regnault has found that, a being the coefficient of expansion per degree Centigrade,

for Carbonic acid gas	$a = \cdot 003710$,
„ Protoxide of Nitrogen	„ $= \cdot 003719$,
„ Sulphurous acid gas	„ $= \cdot 003903$,
„ Cyanogen	„ $= \cdot 003877$;

the last two of which are notably greater than the coefficient of expansion of air; but these are precisely the gases that can be most easily liquefied, while it is found that for all gases which can be liquefied only with great difficulty, a has very nearly the same small value, ·003665, that it has for air. Hence M. Regnault modifies the law of Dalton and Gay-Lussac by saying that the coefficients of expansion of all gases approach more nearly to equality as their intensities of pressure become more feeble; so that it is only when gases are in a state of great tenuity that they have the same coefficient of expansion.

28. General Equation for the Transformation of a Gas. Given the volume, v, of a mass of gas at the temperature t, and pressure intensity p, find its volume at t' and p'.

First let the temperature be altered from t to t', the pressure intensity remaining p; then the volume v becomes u, where

$$u = v\frac{273 + t'}{273 + t},$$

by equation (2) of last Art.

Now keep the temperature constantly equal to t' and alter p to p'; then u becomes v', where

$$v' = u \cdot \frac{p}{p'},$$

by Boyle's law. Hence we have

$$v' = v \frac{273 + t'}{273 + t} \cdot \frac{p}{p'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{or} \quad \frac{v' \cdot p'}{273 + t'} = \frac{v \cdot p}{273 + t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{or} \quad \frac{v' \cdot p'}{T'} = \frac{v \cdot p}{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

where T and T' are the *absolute* temperatures of the gas.

Hence, whatever changes of pressure and temperature may be made in a *given mass* of gas, we have the result

$$\frac{v \cdot p}{T} = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

between its volume, pressure intensity, and absolute temperature.

This most important result is the general equation for the transformation of a given mass of gas.

29. Formula in English Measures. Since the freezing point of water is marked 32° on Fahrenheit's thermometer, and the boiling point 212°, the fractional expansion of gas per degree Fahrenheit is $\frac{\cdot 3665}{180}$, or about $\frac{1}{491 \cdot 13}$, *of the volume* at 32°. This fraction is usually taken as $\frac{1}{492}$; and this, as will presently be seen, would place the absolute zero of temperature 460 Fahrenheit degrees below the zero of the Fahrenheit scale. The experiments of Joule and Thomson indicate $-460 \cdot 66$ as the position of the absolute zero; but for all practical purposes we can take -460.

If a given mass of gas has a volume u at 32° F., and its temperature is raised to t, we have, if the intensity of pressure is unaltered,

$$v = u\left(1 + \frac{t-32}{492}\right).$$

Hence $v = u\,\frac{460+t}{492}$; and if v' is its volume at t', we have

$$\frac{v}{460+t} = \frac{v'}{460+t'}.$$

If the intensity of pressure, estimated in any way, alters from p to p',

$$\frac{vp}{460+t} = \frac{v'p'}{460+t'}. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

It thus appears that if the temperature of the gas were reduced to $-460°$ F., its volume would vanish, supposing that it obeys the laws of a gas during the whole process.

If we denote by T the absolute temperature, $460+t$, of the gas, we have the general equation of transformation of a given mass

$$\frac{vp}{T} = \frac{v'p'}{T'} = \text{constant}. \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

30. Law of Avogadro. One of the fundamental laws of gases is known as the Law of Avogadro. It is the following: *equal volumes of all substances when in the state of perfect gas, and at the same temperature and intensity of pressure, contain the same number of molecules.*

This law enables us to find the relative molecular weights of all substances by converting these substances into vapours, and then measuring the weights of known volumes of the vapours at known temperatures and intensities of pressure. Thus, it is found that a cubic foot of oxygen weighs 16 times as much as a cubic foot of hydrogen under like conditions of temperature and pressure; hence we conclude that the mass of each molecule of oxygen is 16 times that of a molecule of hydrogen.

31. Air Thermometer. A long capillary glass tube, *AC*, terminating in a bulb, *B*, is filled with air, and a short thread, *m*, of mercury is inserted into it, the end of the tube (beyond *C*) being open. In order to fill the bulb and portion of the tube to the left of *m* with air deprived of moisture, the tube and bulb are first filled with mercury which is boiled in the bulb. The open end is then inserted into a cork fitting into the neck of a tube, *D*, filled with chloride of calcium, which has the property of absorbing aqueous vapour from air, a fine platinum wire having been inserted into the stem *CA* through the tube *D*. If the instrument is supported in a position slightly inclined to

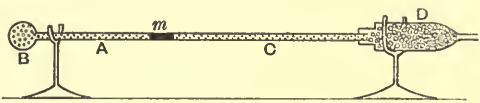


Fig. 54.

the horizon on two stands and the platinum wire is agitated, air enters through the chloride of calcium, and gradually displaces the mercury from the bulb and stem, the process being stopped when only a very short thread of mercury is left.

The air in the instrument may now be considered to be dry.

Detach the stem from the drying tube *D*, and place it in a vertical position with the bulb *B* in a vessel filled with melting ice. Suppose the barometer to stand at 760 mm., thus indicating the standard atmospheric pressure. Then when the air has assumed the temperature of the melting ice, mark 0 on the stem *AC* at the under limit of the mercury index *m*.

If then the instrument is placed vertical with the bulb B surrounded by the steam of boiling water—close to the surface of the water, but not *in* it—the index m will move up towards C, and at its lower limit let 100 be marked on the stem. Thus the two standard Centigrade temperatures are marked on the stem, and the intervening space is to be divided into 100 equal parts if the stem has previously been ascertained to be of uniform bore. The graduations may be carried then below zero and beyond 100°.

If the tube of the air thermometer is made cylindrical all through—so that the bulb B is simply a uniform continuation of the stem—and we continue the graduations to 273 parts below the zero, we shall here reach the bottom, B, of the tube.

Hence the definition of the *absolute temperature* of a body, which we are so far justified in giving, is simply, in the words of Clerk Maxwell, its *temperature reckoned from the bottom of the tube of the air thermometer*.

The upper end of the stem of an air thermometer necessarily remains open to the atmosphere, otherwise the index, m, would not move or would scarcely move at all: if the end were closed and the air uniformly heated, m would not move.

Hence the air thermometer cannot be used to indicate temperature except in conjunction with the barometer. If the latter stands at p instead of p_0, the standard height (which we have above supposed to be 760 mm.) and the temperature indicated by the index m is t, the real reading is not t but that at which the index would stand if the intensity of pressure were altered to p_0. To find the point at which the index would stand in this case, let s be the area of the cross-section of the tube, c the length of the tube between two successive degrees, and B the volume of the bulb and tube up to the zero mark. Then when the index

m stands at the mark t', the volume of the gas is $B + cst'$. But since at the absolute zero the volume of the gas would vanish, $B = 273\,cs$; hence

$$v = (273 + t')\,cs,$$

and this is at the intensity of pressure p, its true temperature being t. If p were altered to p_0 without any change in the true temperature, the index would stand at t and the volume would be $(273 + t)\,cs$. Now since these volumes are inversely as the intensities of pressure, we have

$$273 + t = (273 + t')\frac{p}{p_0},$$

which gives the true reading.

Examples.

1. A circular cone, hollow but of great weight, is lowered into the sea by a rope attached to its vertex; find the volume of the compressed air in the cone when the vertex is at a given depth below the surface.

Let Fig. 55 represent a section of the cone; let c be the depth of the vertex below the surface, LN, of the water, h = height of cone, V = its volume, t = the temperature of the air at the surface, t' = temperature of the water, and therefore of the air in the cone; let P be the surface of the water within the cone, and let k be the height of a column of sea-water in a water barometer.

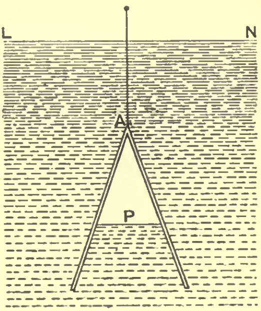
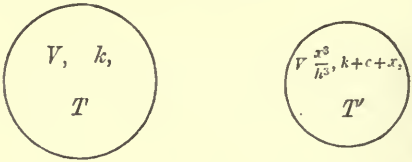


Fig. 55.

If these quantities are in English measure, we may regard the lengths as measured in feet, and the temperature as Fahrenheit; then k will be about 33 feet.

Now if x is the depth of P below A, the volume of the air in the cone is $\frac{Vx^3}{h^3}$. The intensity of pressure of this air measured by a column of water is $k+c+x$. Hence the following diagrams represent the history of this mass of air as regards volume, temperature, and intensity of pressure:



in which T and T' are *absolute* temperatures.

From Art. 28 or Art. 29 we have, then,

$$\frac{Vk}{T}=\frac{Vx^3(k+c+x)}{h^3 T'},$$

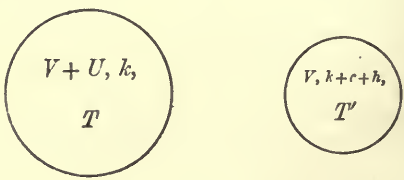
$$\therefore\ x^4+(k+c)\,x^3-kh^3\frac{T'}{T}=0,$$

from which x can be found.

A vessel used in this manner is called a *diving-bell*. The above is a conical diving-bell.

2. If in the above position of the cone it is desired to free the interior of water completely by pumping the air above the surface into the cone, find the volume of this surface air that will be required.

Let U be the volume required, and h the height of the cone; then suppose the cone to be wholly filled with air of the temperature t' of the surrounding water, and write down the history of this air, thus:



$$\therefore\ \frac{(V+U)k}{T}=\frac{V(k+c+h)}{T'},$$

$$\therefore \ U = V\left\{\left(1 + \frac{c+h}{k}\right)\frac{T}{T'} - 1\right\}.$$

(It is not improbable that the student will fall into the error of supposing that U can be calculated as the volume of the surface air which is required to occupy the lower portion of the cone in Fig. 55, i.e. the portion occupied by water.)

Of course the result is the same whether the vessel is conical or of any other figure.

3. If a conical diving-bell of height h feet contains a mercurial barometer the column of which stands at p_0 inches when the bell is above the surface of the water, and at a height p when below, infer the depth of the top of the bell below the surface.

Result. $\frac{13 \cdot 596}{12}(p - p_0) - h\left(\frac{T' p_0}{Tp}\right)^{\frac{1}{3}}$ feet.

4. Deduce the depth for a cylindrical or prismatic bell.

Result. $\frac{13 \cdot 596}{12}(p - p_0) - h\frac{T' p_0}{Tp}$.

5. A cylindrical diving-bell 12 feet high is lowered into water until the depth of the top of the bell is 60 feet; the height of a mercury barometer at the surface is 30 inches and the temperature of the air 76° F.; the temperature below is 48°; find the height to which the water rises in the bell.

Result. 8·1 feet.

What volume of surface air must be forced down into the bell to expel all the water?

Result. 2·29 times the volume of the bell.

6. If the bell is 10 feet high and is lowered until the depth of the top is 40 feet, find the same things when the surface temperature is 60° and the temperature below is 50°, the height of a water barometer being 34 feet.

Result. 4·26 feet; 1·52 × vol. of bell.

7. Find the tension of the suspending chain in a diving-bell which occupies any position in water.

Result. The weight of the bell and its appurtenances diminished by the weight of the water which is displaced from all causes.

(The water is displaced by the chain, the thickness of the bell, and the air within the bell; the weight of this water is the force of buoyancy. In strictness, the weight of the contained air should be added to that of the bell.)

8. If at the bottom of a river 40 feet deep, when the temperature is 40° F., a bubble of air has the volume $\frac{1}{10^5}$ of a cubic inch, what will be its volume on reaching the surface where the temperature is 50° F., and the height of a water barometer is 34 feet?

Ans. $\frac{2 \cdot 22}{10^5}$ cubic inches.

9. If an open vessel (such as a tumbler) made of a substance whose specific gravity is greater than that of water is forced, mouth downwards, into water, show that its equilibrium becomes unstable after a certain depth has been reached.

(If the volume of the solid substance of the vessel is v, and in any position of the vessel if X is the volume of its compressed air, the downward force, P, required to hold it in equilibrium is given by the equation

$$P = Xw - v(w' - w),$$

where w = specific weight of water, w' = specific weight of substance of vessel.

Hence when X is so far diminished by forcing the vessel down that $Xw = v(w' - w)$, the pressure P vanishes, and after this an upward pull would be required.)

10. If v is small (i. e., if the thickness of the vessel is small), and if V is the volume of the interior of the vessel, prove that when the position of instability is reached, the depth of the top of the vessel below the surface of the water is approximately

$$k\left\{\frac{Vw}{v(w' - w)} - 1\right\},$$

where k is the height of a water barometer at the surface.

11. In a mercury barometer tube the height of the mercury column is 30 inches, the area of the cross-section of the tube is 1 sq. inch, and the vacant space is 4 inches long. If $\frac{1}{8}$ of a cubic inch of the external air is passed up into the tube, what depression of the column will it produce?

Ans. 1 inch.

12. If v cubic inches of the external air at the absolute temperature T are inserted into the Torricellian vacuum of a uniform cylindrical barometer tube, calculate the depression produced in the column of mercury if the absolute temperature changes to T'.

Let h inches be the height of the barometer at first, $a =$ length of Torricellian vacuum, s square inches $=$ area of cross-section of tube, $x =$ length of tube finally occupied by the air; then

$$x(x-a) = \frac{T'}{T} \cdot \frac{vh}{s}.$$

13. A diving-bell of any shape occupies a given position below the surface of water; the bell has a platform inside; if a large block of wood falls from the platform into the water, prove that the water will rise inside the bell, but that the bell now contains less water than before.

Let the depth of the top be c, let h be the height of a water barometer at the surface, put $k = c + h$, let $B =$ volume of the block of wood, w' its specific weight, $w =$ specific weight of water, $V =$ whole volume of the interior of the bell, let x be the depth of the water in the bell below the top of the bell, and let X be the volume of the interior of the bell above this surface. Then

$$(X - B)(x + k) = Vh. \quad \text{. . . . . . . } (1)$$

When the wood falls the volume of it which remains above the surface is $B\left(1 - \frac{w'}{w}\right)$. Let x' be the new depth of the water surface in the bell below the top of the bell, and X' the volume of the interior of the bell above this new surface. Then

$$\left\{X' - B\left(1 - \frac{w'}{w}\right)\right\}(x' + k) = Vh. \quad \text{. . . . . } (2)$$

Now since X obviously increases with x, we must have $x' < x$, since in the opposite case each of the factors at the left-hand side of (2) would be greater than the corresponding factor in (1), and the equations would be inconsistent.

Hence the surface of the water rises.

Again, if $\Omega =$ the first volume of water in the bell, and Ω' the second

$$\Omega = V - X,$$

$$\Omega' = V - X' - B\,\frac{w'}{w};$$

$$\therefore\ \Omega - \Omega' = B\frac{w'}{w} - (X - X'). \quad . \quad . \quad . \quad . \quad (3)$$

Now from (1) and (2)

$$X - X' = B\,\frac{w'}{w} - \frac{Vh\,(x - x')}{(x + k)\,(x' + k)},$$

$$\therefore\ \Omega - \Omega' = V\,\frac{h\,(x - x')}{(x + k)\,(x' + k)},$$

which shows that Ω' is less than Ω.

32. Weight of Gas. It is obvious that the weight of a cubic foot of air, or any other gas, is not the same when its temperature is 20° or 100°, as when it is 0°, supposing the intensity of pressure the same. In other words, the weight of a cubic foot of air depends on the temperature and pressure intensity at which it is taken.

Taking the units of the Metric System, let us inquire what is the weight of v litres (i.e. cubic decimetres) of dry air when its temperature is $t°$ C. and its intensity of pressure denoted by a column of mercury p millimetres high.

Supposing that we knew the weight of 1 litre of air when its temperature is 0° and its intensity of pressure that of a standard atmosphere, denoted by a column of mercury 760 mm. high, we could answer the question by finding the number of litres which would be occupied by the given v litres if its state were changed from (p, t) to $(760, 0°)$.

But by (1) or (2) of Art. 28, if we put $t' = 0$, $p' = 760$, we have

$$v_0 = \frac{273}{760} \cdot \frac{vp}{T} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now M. Regnault found that the mass of

1 litre of dry air at (760, 0°) = 1·293187 grammes . (2)

Hence the mass of v litres at (p, t) is v_0 multiplied by this number. Denoting the mass in grammes by W, we have then

$$W = \cdot 4645 \frac{v \cdot p}{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

in which, be it remembered, T is the absolute Centigrade temperature of the air, p its pressure intensity estimated in millimetres of mercury, v its volume in litres, and W its mass in grammes.

For any other gas, if its specific gravity at (760, 0°) is denoted by s, the mass of a litre of it in this state is $1{\cdot}293187 \times s$ grammes, and evidently if W is the mass of v at (p, t), we have simply

$$W = \cdot 4645 \frac{v \cdot p \cdot s}{T} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

The specific gravity of a gas is above assumed to be the ratio of the weight of any volume of the gas to the weight of an equal volume of dry air at (760, 0°); but it is easy to see that we get exactly the same result by taking the ratio of the weight of a volume of the gas at (p, t) to the weight of an equal volume of air also at (p, t), whatever the pressure intensity, p, and the temperature, t, may be, if it be true that all gases have the same coefficient of expansion; for, equal volumes, v, of the two gases at (p, t) will become equal volumes, v_0, at (760, 0°) since

$$v_0 = \frac{v \cdot p}{1 + at} \cdot \frac{1}{760},$$

and a is the same for both gases.

33. The equation $p = k\rho$. From (β) we see that if for $\frac{W}{v}$ we write ρ, where ρ is the mass, in grammes, of the gas per litre, we have

$$p = \frac{1}{\cdot 4645} \cdot \frac{T}{s} \cdot \rho, \quad \ldots \quad (1)$$

p being measured in millimetres of mercury.

Let p be measured in grammes' weight per square centimetre, and let ρ be the mass, in grammes, per cubic centimetre; then, if we have 1 cubic cm. of the gas at (p, t), and this becomes x cubic cm. at zero and an intensity of pressure of $76 \times 13{\cdot}596$ grammes' weight per sq. cm., we have

$$\frac{x \times 76 \times 13{\cdot}596}{273} = \frac{1 \times p}{T} \quad \ldots \quad (2)$$

Now the mass of 1 cubic cm. at the latter temperature and pressure being $\frac{1{\cdot}293187}{1000} s$ grammes, the mass of x cubic cm. is obtained by multiplying this by the value of x in (2), and this mass is ρ, the density of the gas at (t, p), i.e. the mass of 1 cubic cm. Hence we have

$$p = 2926{\cdot}9 \frac{T}{s} \rho, \quad \ldots \quad (3)$$

where p is in grammes' weight per sq. cm. and ρ in grammes per cub. cm. Hence if we write the relation between p and ρ in the form $p = k\rho$, we see that

$$k = 2926{\cdot}9 \frac{T}{s}.$$

If p is measured in dynes per sq. cm., we must multiply this value of k by the value of g in dynes, i.e. by 981 (about). In this case, then,

$$p = 2926{\cdot}9 \frac{gT}{s} \rho. \quad \ldots \quad (4)$$

(Observe that here s is the specific gravity of the gas referred to air.)

If v is the volume of the gas at (p, T), and w its mass, we have, by multiplying both sides of (3) by v,

$$pv = 2926{\cdot}9 \frac{w}{s} . T. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It is sometimes useful to express p in kilogrammes' weight per square decimetre, v in cubic decimetres, and w in kilogrammes ; in which case (5) becomes

$$pv = 292{\cdot}69 \frac{w}{s} . T. \quad . \quad . \quad . \quad . \quad (6)$$

It is usual to write (5) or (6) in the form

$$pv = RwT, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where R stands for the constant $\dfrac{2926{\cdot}9}{s}$ in the first case.

34. Formulae in English Measures. The equation connecting the volumes, &c., of a given mass of gas in English measures is

$$\frac{vp}{460+t} = \frac{v'p'}{460+t'} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To obtain the formula for the mass of air, analogous to (a), Art. 32, we may either convert the metric formula into English measures, or deduce a formula from special observations on the mass of a given volume of air under standard conditions. Dr. Prout found that the mass of 100 cubic inches of dry air at the temperature 60° F. at an intensity of pressure indicated by 30 inches of mercury in a barometer tube is 31·0117 grains ; in other words,

the mass of 1 cubic foot at (60°, 30″) is ·0765546 pounds. (a)

Now if we have v cubic feet of dry air at $(t°, p)$, where p is in inches of mercury, this would, by (1), become

$$\frac{520}{30} . \frac{vp}{460+t}$$

cubic feet at (60°, 30), and multiplying this by the number (a), we have

$$W = 1{\cdot}327 \frac{vp}{460+t}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

for the mass, in pounds, of the given v cubic feet at (t°, p), the intensity of pressure, p, being supposed taken in inches of mercury.

For a gas of specific gravity s (referred to air),

$$W = 1{\cdot}327 \frac{vps}{460+t}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If p is estimated in pounds' weight per square foot, and $460+t$ is denoted by T, we have, with sufficient accuracy,

$$W = \frac{1}{53{\cdot}3} \frac{vps}{T},$$

$$\therefore p = 53{\cdot}3 \frac{T}{s} \rho,$$

where ρ is the density of the gas in pounds per cubic foot, p its intensity of pressure in pounds' weight per square foot, &c.

To obtain the analogue of (6), Art. 33, multiply both sides by v; then

$$pv = 53{\cdot}3 \frac{w}{s} T,$$

where w is the mass of the gas in pounds; or

$$pv = RwT,$$

where R stands for $\frac{53{\cdot}3}{s}$.

Examples.

1. If a cubic inch of water is converted into steam at 212° F., find the volume of the steam.

Result. 1696 cubic inches. Hence it is roughly true that a cubic inch of water yields a cubic foot of steam.

2. Calculate the mass of air in a room whose dimensions are 18, 18, and 10 feet, the temperature being 60° F. and the barometer standing at 30 inches.

Result. 248 pounds.

3. If 1 pound of water is converted into steam at 212° F. at an intensity of pressure of 15 pounds' weight per square inch, find the volume of steam in cubic feet.

Result. 26·66.

4. If any volume of water is converted into steam at t° F. under a pressure of p pounds' weight per square inch, show that the ratio of the volume of steam formed to that of the water is about

$$37{\cdot}1\,\frac{460+t}{p}.$$

(This is called the *relative volume* of steam at the temperature t.)

35. Barometric Formula. We are now in a position to deduce a formula for the height of a mountain, by assuming the temperature of the air to be constant within these limits. The latter assumption would often be far from the truth, but we shall presently see how it can be corrected.

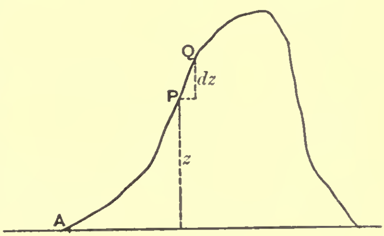


Fig. 56.

Let A, Fig. 56, be a point at the base of the mountain where the height of the barometer is p_0 mm.; let P be a point at a height of z decimetres above A, and at P let the barometric height be p mm.; let Q be a point close to P at a height dz above P; let t be the temperature of the air at P.

Imagine a horizontal area of 1 square decimetre at P; then the atmospheric pressure on this area is the weight of the column of air standing on it and extending to the upper limit of the atmosphere. Hence the difference of the pressures on this area at P and Q is the weight of the vertical column of air of height dz standing on 1 square decimetre. But if the height of the barometer at Q is $p+dp$ mm., the pressure on this area is also the weight of a column of mercury $-dp$ millimetres high. Hence weight of $-10\,dp$ c.c. of mercury = weight of dz litres of air; that is (p. 121)

$$-135\cdot96\,dp = \cdot4645\,\frac{pdz}{273+t}, \quad . \quad . \quad . \quad (1)$$

which gives the relation between dp and dz. The integral gives, if t is constant,

$$z = 673\cdot962\,(273+t)\log_{10}\frac{p_0}{p},$$

z being in decimetres. If z is taken in metres, this becomes with sufficient accuracy

$$z = 18400\left(1+\frac{t}{273}\right)\log_{10}\frac{p_0}{p}. \quad . \quad . \quad . \quad (2)$$

If the variation of gravity is taken into account, let p be measured in grammes' weight per square centimetre, since it matters not in what units p and p_0 are measured in (2). Then if r denotes the radius of the Earth,

$$\frac{dp}{dz} = -\rho\,\frac{r^2}{(r+z)^2},$$

$$p = 2926\cdot9\,T\rho,$$

$$\therefore\ r^2\left(\frac{1}{r+z_0}-\frac{1}{r+z}\right) = 673\cdot962\,T\log_{10}\frac{p_0}{p},$$

in which again we can put

$$\frac{p_0}{p} = \frac{h_0}{h}\left(\frac{r+z}{r+z_0}\right)^2,$$

where h_0 and h are the observed barometric heights at the two stations. Neglecting $\frac{z^2}{r^2}$, and taking z in metres, this equation becomes

$$z-z_0 = 18400\left(1+\frac{t}{273}\right)\log_{10}\frac{h_0}{h}\left(1+2\frac{z-z_0}{r}\right).$$

Assuming $r = 637 \times 10^4$ metres, and observing that

$$\log_{10}\left(1+2\frac{z-z_0}{r}\right) = \cdot 8686\,\frac{z-z_0}{r},$$

approximately, this equation gives

$$z-z_0 = 18400\left(1+\frac{t}{273}\right)\left\{1+\cdot 0025\left(1+\frac{t}{273}\right)\right\}\log_{10}\frac{h_0}{h}.$$

The barometric readings p_0, p in these equations require slight corrections for temperature; for we have assumed that the mass of a cubic centimetre is 13·596 grammes, which is true only if the temperature is zero.

Since the coefficient of expansion of mercury per degree C. is $\frac{1}{5550}$, a volume which was 1 c.c. at 0° would become $1+\frac{t}{5550}$ c.c. at $t°$ while its mass remains 13·596; hence at the temperature t the mass of 1 c.c. would be

$$13\cdot 596\left(1-\frac{t}{5550}\right)\text{ grammes,}$$

nearly, and the mass of

$$10\,dp\text{ c.c.} = 135\cdot 96\left(1-\frac{t}{5550}\right)dp;$$

so that we must regard the height p observed at the temperature t as *corrected*, or reduced, to $\left(1-\frac{t}{5550}\right)p$, and hence at the base, where the temperature is t_0 and the *observed* barometric height is p_0, we must replace p_0 by $\left(1 - \frac{t_0}{5550}\right)p_0$, and at the top similarly use $\left(1 - \frac{t}{5550}\right)p$ instead of p. These are called the *corrected barometric heights.* In the term $1 + \frac{t}{273}$ which occurs in (2) it is usual to take t as the mean of the temperatures at the top and bottom.

If the Fahrenheit thermometer is used, the coefficient of expansion of mercury may be taken as $\frac{1}{10000}$, and the corrected height is $\left(1 - \frac{t-32}{10000}\right)p$.

The formula in English measures which corresponds to (2) is found in the same way to be

$$z \text{ (feet)} = 122{\cdot}73\ (460 + t) \log_{10} \frac{h_1}{h}, \quad . \quad . \quad (3)$$

where t is mean Fahrenheit temperature.

EXAMPLES.

1. The height of the barometer on the ground being 30 inches and the temperature 32° F., what height corresponds to a fall of 1 inch in the barometric height?

Ans. About 890 feet.

2. The temperature of the air being assumed at 0° C. and the barometric height at the base of a mountain 76 cm., if the mountain is 4600 metres high, what will be the barometric height at the top?

Ans. 42·74 cm.

3. At the foot of a mountain the temperature of the air is 66° F., and the height of the barometer 29·35 inches; at the top the temperature is 50° F., and the barometric height 24·81; find the height of the mountain, assuming the coefficient of expansion of mercury to be $\frac{1}{10000}$ per degree F.

About 4590 feet.

CHAPTER VI

HYDRAULIC AND PNEUMATIC MACHINES

36. Water Pumps. *The Common Pump.* Let DB, Fig. 57, represent a vertical section of an iron cylinder terminating in a much narrower cylinder or pipe, BA, which dips into a well from which water is to be raised. In the cylinder DB, or barrel, works a piston having a valve, v, which opens upwards, the piston rod, r, being connected at c with a lever, fH, working about the fulcrum f.

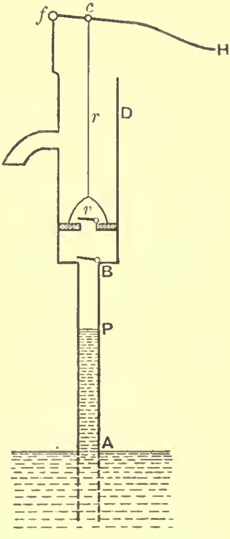


Fig. 57.

At the top, B, of the suction pipe is a valve which also opens upwards.

Let the piston be at the bottom of the barrel, the level of the water in the well being A, and the pipe and barrel both completely filled with air.

When the piston is raised by means of the handle, H, of the lever, the valve v remains closed, and the pressure of the air in the pipe opens the valve at B, the air in AB rising and partly filling the barrel, its intensity of pressure also diminishing. As a result of this diminution of pressure below its original (atmospheric) value, the atmospheric pressure on the water forces some of the liquid into the pipe. Let P be the level of the water in the pipe at the end of the upward stroke of the piston. On the downward stroke of the piston the valve B closes and v opens allowing the air in the barrel—which the downward motion of the piston tends to compress—to escape through the piston into the atmosphere, until the piston again reaches the bottom of the barrel. On again raising the piston, the valve v closes and that at B opens, thus allowing the air in BP to expand and to diminish its intensity of pressure; and, in consequence, more water is forced up into the pipe, and perhaps into the barrel. This process being continued, the water ultimately reaches the level of the spout and is thus driven out.

To find the height to which the water is raised in the pipe by the first stroke of the piston, let $A =$ area of cross-section of barrel, $a =$ area of cross-section of pipe, $l =$ length of stroke of piston, $c =$ length BA, $h =$ height of a water barometer, and $AP = x$. The volume of air in the suction-pipe before the stroke is ac, and its intensity of pressure is represented by h. At the end of the stroke the volume of this mass is $a(c-x)+lA$, and its intensity of pressure is represented by $h-x$. Hence

$$(h-x)\{a(c-x)+lA\} = ach,$$

$$\therefore\ ax^2-(ac+ah+lA)x+lhA = 0,$$

which determines x.

When the water is flowing out of the spout, there will be a tension in the piston rod on an upward stroke, which is found thus. Let z be the vertical height of the piston above the level A, and let z' be the height of the column of water (which we may suppose to reach to any point, D, of the barrel) above the piston. The valve at B being open, there is a continuous water communication between A and the bottom of the piston. Hence if w is the weight of a unit volume of water, the intensity of the upward pressure exerted by the water on the bottom of the piston is $w(h-z)$; and the intensity of downward pressure exerted on the top of the piston is $w(h+z')$; therefore the total downward pressure on the piston is $w(z+z')A$. This is equal to T, the tension of the rod, if we neglect any acceleration of the piston. Hence, approximately,

$$T = w \,.\, A \,.\, DA,$$

which shows that the tension of the rod is the weight of a vertical column of water having the piston for base, and for height the difference of level of the water in the barrel and that in the well.

On the downward stroke there is a *pressure* in the rod, which is approximately equal to the weight of the column of water above the piston.

When the water is flowing out, the force required at H to work the piston on the upward stroke is $T \times \frac{fc}{fH}$, where T has the above value.

It is obvious that, for the working of the pump, the length AB of the suction-pipe above the well must be less than the height of a water barometer, i.e., about 34 feet; and, owing to imperfect fittings, AB must be considerably less than this—say about 25 feet.

In the middle ages a curious modification of the common pump, called the *bellows pump*, was employed in Europe. Instead of a piston worked by a lever, fH (Fig. 57), a large bellows was attached firmly to the top of the barrel, and the nozzle of the bellows was the spout through which the water was forced.

The top of the barrel fitted into the interior of the bellows through a hole in the lower board of the bellows; there was no valve in the top board, but there was one opening outwards fixed in the nozzle. The action was, of course, the same as that in our modern pump.

The Forcing Pump. This is an instrument for raising water to a great height. It differs from the previous pump in having a completely solid piston.

To the barrel of the pump is attached the pipe through which the water is to be raised. This delivery pipe is provided with a valve at D, Fig. 58, opening upwards, and this pipe is represented as being provided with a spout and stop-cock by means of which the machine can be made to act as a Common pump.

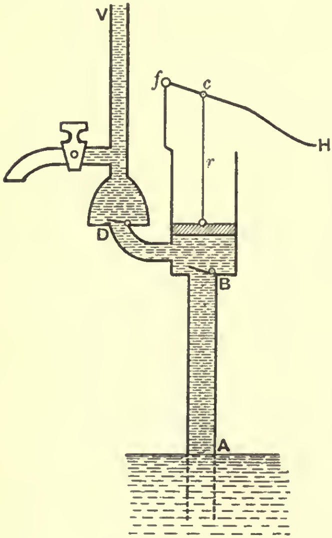


Fig. 58.

The action is the same as in the previous case.

On the downward stroke of the piston, the valve at B closes and that at D opens, and through this latter the water is forced out of the barrel into the delivery pipe, DV.

There is then a pressure in the piston-rod, r, equal to the weight of a column of water having the piston for base, and for height the difference of level between the piston and the water, V, in the delivery tube.

On the upward stroke there is a tension in the rod, whose value is the same as in the previous pump.

The Fire Engine. This is simply a double forcing pump. The figure (Fig. 59) represents the two barrels, P and Q, as immersed in a tank, DEI, full of water; and from this tank the pumps, which are both worked by the lever AB, force the water through a hose connected with the chamber C at h. The water is forced through this hose to the place where the fire is to be extinguished. The action of the valves is obvious in the figure. Such would be the arrangement in a small fire engine, the tank DEI being filled by buckets of water brought by hand.

When large fires have to be dealt with, a supply thus derived would be of no use, and the water which is pumped through the hose must be derived from a well or other reservoir by means of a suction pipe. The figure represents at *c* the place where such a pipe can be attached to the engine.

The chamber *C* is partly filled with water and partly with air, and is called an *air chamber*. Such a chamber may be, and often is, fitted to forcing and other pumps, the object being to render the stream of water from the

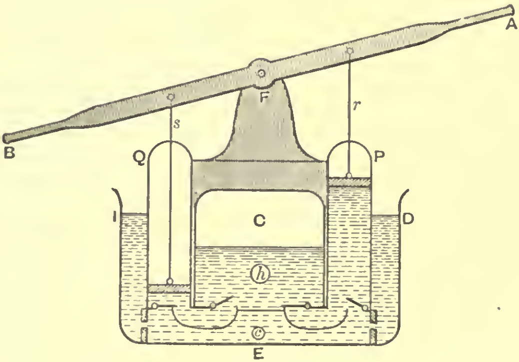


Fig. 59.

delivery pipe at *h* continuous instead of intermittent; and this result is evidently secured by means of the compressed air at the top of the chamber; for, since this air was originally at the atmospheric pressure (when it filled the whole of the chamber), its intensity of pressure after the chamber is partly filled with water is greater than this value. This increased pressure is therefore continuously exerted on the top of the water in the chamber and helps to drive the stream through the hose.

The Hydraulic Screw. This is one of the most ancient machines for raising water, and is still employed in some countries. It is often called the *Screw of Archimedes*, because its invention is supposed to be due to the philosopher of Syracuse. There is, however, reason to think that it was first employed in Egypt.

As represented in Fig. 60, it consists of a pipe wound spirally on an axis which is fixed in a position inclined to the vertical, its extremities fitting into two solid supports,

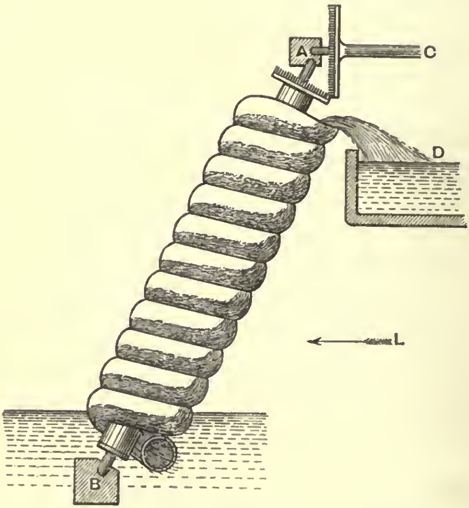


Fig. 60.

A, *B*, the axis (and, with it, the screw) being able to revolve freely. Both ends of the spiral pipe are open, and the lower is immersed in the water which is to be raised to the level *D*.

The revolution of the screw can be effected in various ways: in the figure it is represented as produced by the revolution of a shaft *C* fitted with a toothed wheel which gears with another fitted to the top of the axis of the screw.

When the screw is made to revolve so that the lower end comes towards us in the figure, water entering at this end continually drops to the lowest point of each part of the spiral, and is thus carried continuously up to the top where it is discharged.

There is a certain condition that must be satisfied by the inclination of the axis of the screw to the vertical and the angle of the spiral in order that the machine may be able to raise the water. The condition is this : *the inclination of the axis of the screw to the horizon must be less than the inclination of the tangent line of the spiral to the axis of the screw.* To prove this, we may put the matter thus: *the axis of the screw must be so much inclined to the vertical that it is possible to draw a horizontal tangent to the spiral.* This is obvious, because if we imagine a single particle (suppose a small marble) entering the lower end of the pipe, it would not drop down farther through the opening unless there were in the pipe a place in which the particle could rest under the action of its own weight and the reaction of the pipe on it; and at such a place the tangent to the spiral must be horizontal.

The Hydraulic Screw is capable of a *differential* form. Suppose the screw in Fig. 60 not to dip into water at its lower extremity, but to receive into the upper end of the pipe at A a stream of water from any source. Then the screw, being fixed exactly as represented, would be driven by this stream in the direction opposite to that in which it was caused to rotate under the previous circumstances.

Now suppose that it is, as before, desired to raise water from a well at the extremity B to a position D, and suppose that there is available a stream of water at some lower level, represented by the arrow L. Let a second pipe be spirally wound round the axis BA outside the tube represented in the figure *and in the reverse sense*, its upper end terminating at the level L. If the stream L is led into the upper end of this second pipe, it will cause the whole machine to rotate in the sense required to raise the water from the well by means of the internal spiral pipe.

The idea of the Differential Hydraulic Screw appears to be due to the ingenious Marquis of Worcester, who published his notions on this machine and on many others in a work called *A Century of Inventions*, in the reign of Charles II.

The Hydraulic Ram. This is a machine in which the momentum of a stream of water is suddenly stopped, with the result that a portion of the water is forced up to a considerable height.

Let *AB*, Fig. 61, be a stout iron chamber to which

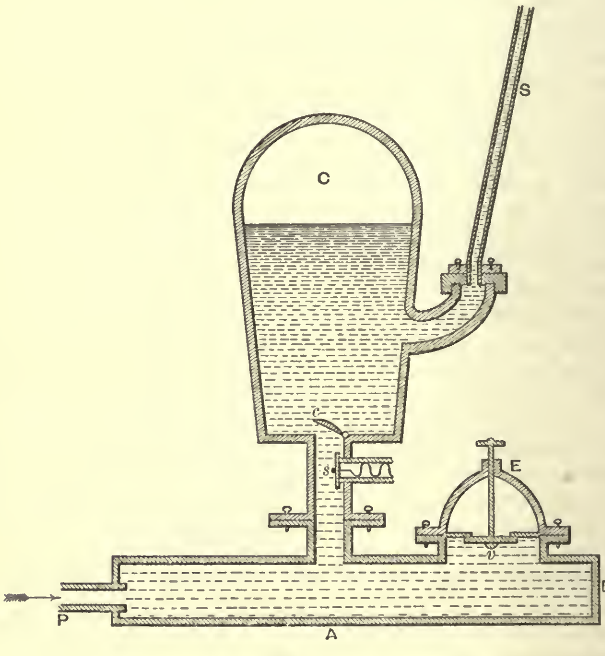


Fig. 61.

is attached a pipe *P* which admits a flow of water from a stream or reservoir the level of which may be only slightly higher than that of *AB*. The vessel *AB* is fitted with a support, *E*, for a valve, *v*, which can move freely up and down. When this valve falls, there is a free communication between the interior of *AB* and the atmosphere, and if AB is full of water, some of this water flows away through the opening at v, and is wasted.

To the top of the vessel AB is screwed a chamber, C, which has a valve c opening upwards, and also a valve s opening inwards. This latter valve is attached to a rather weak spring fixed to a side pipe opening into the chamber C. Finally, a supply pipe, S, is attached to the chamber C.

Imagine the whole machine free of water, so that the valves c and v are down; and then let the stream flow in at P. At first the water will rush through the opening at v; but soon the rush of water will close this valve, and at this instant, the water being suddenly checked, some will be forced through the opening at c. This valve will then close and v will drop, allowing an outflow again from the vessel AB. The same process will be again and again repeated until the water forced into C rises in the pipe S. The upper part of the chamber C contains imprisoned air, the pressure of which serves to keep the flow up the pipe S continuous. The valve v falling and rising thus regularly, the machine is self-acting. After a long time the air in the air chamber C would be absorbed by the water, and thus the advantage of an air chamber C would be lost. The object of the *snifting valve* s is to prevent this, and the air is renewed as follows. When the valve v drops, the intensity of pressure immediately under it is that of the atmosphere, and therefore the intensity of pressure of the water at s is less than that of the outside air, so that (as the spring which closes s is a weak one) this valve s is forced inwards, thus allowing some air to enter the water; and this air when the valve c is next opened will rise to the top of C and replace the air absorbed by the water.

(It must not be supposed that the forcing of water by this self-acting machine to a height vastly greater than that of the source whence the water was derived involves any contradiction of the principle of Work and Energy; for, it is by means of the kinetic energy generated by gravity in the very large mass of water which flows into and out of the ram that the comparatively small mass of water is raised in the pipe.)

The Hydraulic Ram was invented in 1772 by Whitehurst of Derby, his machine, however, not being self-acting. With him, instead of the self-acting valve *v*, there was a stop-cock through which the water flowed; and it was on the sudden closing of this stop-cock that the water was forced through the valve *c*.

37. Air Pumps. *The Common Air Pump.* In Fig. 62, *B* is a cylinder or barrel in which works a piston with a valve, *c*, opening upwards. The barrel is screwed, or otherwise firmly attached, to a plate, *D*, through which runs a groove which communicates with the interior of the barrel through an opening which can be closed by a valve *a*; the other end, *n*, of this groove opens up through a large plate *PQ*, the upper surface of which is perfectly flat. On this plate is placed a large glass vessel, *A*, called the *receiver*, the mouth of which rests on the plate *PQ*; the rim of the receiver is ground, and it fits the plate so accurately that the junction is air-tight, especially as a layer of grease is rubbed on the rim before it is placed on the plate.

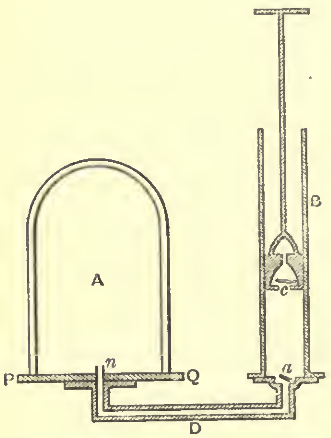


Fig. 62.

The object is to remove the air, partially or completely, from the receiver, and therefore from any body or vessel that may be placed under it.

The manner in which this is effected is obvious from the figure. Suppose the piston in the lowest position in the barrel; then when it is raised, a vacuum tends to form above a, so that the air of the receiver raises this valve and fills the barrel at the end of the stroke. On the descent of the piston, a closes and c opens, and the air in the barrel is thus expelled into the atmosphere. This process is repeated many times, with the result that the air in A is greatly diminished in mass.

To calculate the degree of exhaustion after n strokes of the piston, let the volumes of the receiver and barrel be A and B; let the original intensity of pressure of the air in A be p_0, and let $p_1, p_2, p_3 \dots$ be the intensities after 1, 2, 3, ... strokes.

Then after the piston has been raised the first time the mass of air whose volume and intensity of pressure were (A, p_0) becomes $(A+B, p_1)$; hence

$$(A+B)\,p_1 = Ap_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When the down stroke is ended there is a different mass of air in A, and it is denoted by (A, p_1); and this same mass of air is denoted by $(A+B, p_2)$ at the end of the second upward stroke; hence

$$(A+B)\,p_2 = Ap_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Similarly

$$(A+B)\,p_3 = Ap_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

. . . . .

. . . . .

Hence, by multiplication,

$$(A+B)^n \cdot p_n = A^n \cdot p_0,$$

$$\therefore\ p_n = p_0 \left(\frac{A}{A+B}\right)^n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

which gives the final intensity of pressure. If W_n is the weight of the air finally left, and W_0 the original weight, we have from (*a*) of Art. 32,

$$W_n = W_0 \left(\frac{A}{A+B}\right)^n ,$$

and a similar relation between the final density, ρ_n, and the original ρ_0.

For the very high exhaustions required in the globes of incandescent electric lamps, and in the interior of vacuum tubes, an air pump of this kind would be quite insufficient, because, after the exhaustion has reached a certain limit, the pressure of the gas is insufficient to raise the valve *a*.

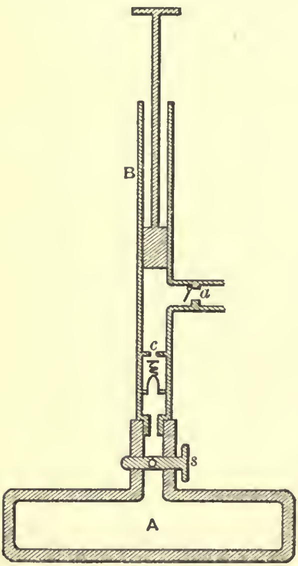


Fig. 63.

Condensing Air Pump. When it is desired to fill a vessel, *A*, Fig. 63, with air or any other gas at a given intensity of pressure, a condensing pump is employed. This machine consists of a cylinder or barrel, fitted with a solid piston, and having a valve, *c*, opening downwards. At the side of the barrel there is attached a pipe having a valve, *a*, opening inwards. When any other gas than air is to be forced into *A*, the vessel supplying this gas is attached to the pipe at *a*. The valve at *a* opens while the piston is raised, and the barrel is filled with the gas. When the piston is lowered, *a* closes, *c* is forced open, and if the stop-cock, *s*, fitted to the vessel *A* is properly turned, the gas enters *A*. On the upward stroke of the piston, c closes, a opens, and the process is repeated.

Let A and B denote the volumes of the chamber A and the barrel, and consider the gas which fills A after n strokes of the piston. Let p be its intensity of pressure, and let p_0 be the intensity of pressure of the gas which fills the barrel: if A is being filled with atmospheric air, p_0 is the atmospheric intensity.

Then the gas whose volume and intensity of pressure are (A, p) was once represented by

$$(A + n B, p_0),$$

supposing that A contained the gas originally. Then

$$A . p = (A + n B) p_0,$$

$$\therefore p = \left(1 + n \frac{B}{A}\right) p_0.$$

The Geissler Pump. When very high exhaustions are required, air pumps with valves and pistons are replaced by pumps in which a column of mercury plays the part of a piston. Of the latter kind is that represented in Fig. 64.

To a certain extent, all air pumps are identical in principle. In each of them a given mass of gas occupying a volume V is made to occupy a larger volume, $V + U$, and then the portion occupying U is mechanically expelled. If this process could be repeated indefinitely, an exhaustion of any degree desired could be obtained; but we have seen that the raising of valves in the common air pumps puts a limit to the process.

The mercury pumps of Geissler and Sprengel are free from this drawback.

AB is a glass tube of greater length than the height of the mercury barometer, having part of the Torricellian vacuum enlarged into a chamber, A, of large capacity.

Above *A* is inserted a two-way stop-cock, *a*, which in the position represented establishes a communication between the chamber *A* and a side tube, *f*, fitted to the

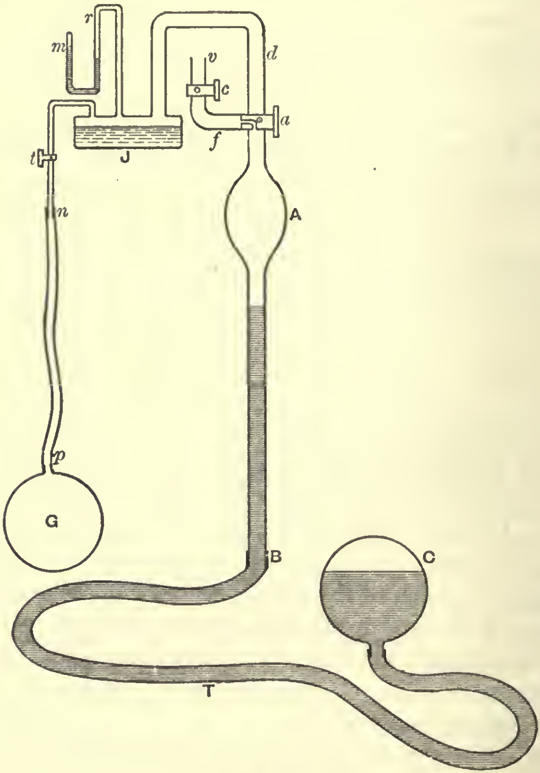


Fig. 64.

tube *BAd*. This tube *f* has a stop-cock, *c*, which in the position represented closes *f*; but if *c* is turned through a right angle it will establish a communication between *f* and the external atmosphere at *v*. If *a* is turned through a right angle from its present position, it will close the communication of A with f, and open one between A and a vessel J to which the portion d is joined as represented. The vessel J contains some sulphuric acid the object of which is to remove aqueous vapour from air which may pass over it; and, by means of a stop-cock, t, J communicates with a very stout indiarubber tube, np, which is connected with the vessel G which is to be exhausted.

To the vessel J is connected a truncated manometer, that is, a bent glass tube, mr, containing mercury which when the air in J is at atmospheric pressure quite fills the leg m. If the air in J is completely removed, the columns of mercury in the legs m and r will assume exactly the same level.

To the end B of the tube BA is attached a stout flexible tube T, which is also fixed to a large reservoir, C, of mercury.

Suppose now that c is turned so as to establish communication between f and the atmosphere at v, and that a is in the position represented (i. e., closing communication of A with d); and let C be raised until the surface of the mercury in BA reaches the stop-cock a, thus expelling all the air from A through fv. Then turn a and t so as to admit air from G through J and d, and lower C to its original position. The air of G now occupies the volume $G + A$ together with the volumes of the communicating tubes.

Turn t so as to break communication with the rarefied air in G, and then turn a so as to establish communication between A and the atmosphere at v. Again, raise C until the mercury in BA drives out the gas from A into the atmosphere; and repeat the process of establishing communication with G, &c. In this way, by repeated operations, the air in G is exhausted almost completely. By this pump the air left in G can be reduced to an intensity of pressure represented by only $\frac{1}{10}$ of a millimetre of mercury.

The Sprengel Pump. Fig. 65 represents this pump, in which, as in the Geissler, the vessel, G, to be exhausted is made part of the Torricellian vacuum of a barometer tube, H.

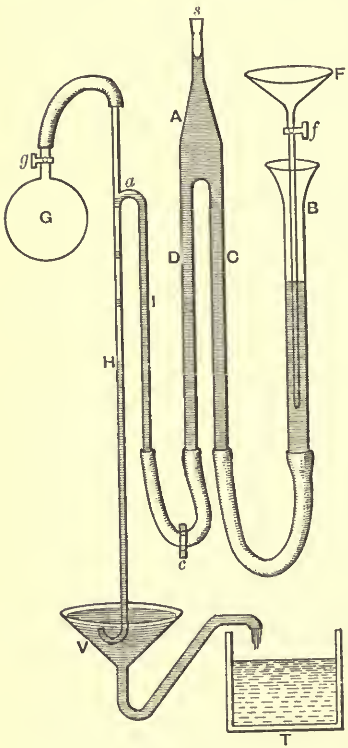


Fig. 65.

A funnel, F, prolonged into a narrow tube fitted with a stop-cock, f, is supported in a vertical position (support not represented in figure) and dips into a wide tube, B, also supported. B is connected by indiarubber tubing with a narrow vertical tube, C, above which is a large chamber, A, open at the top, and fitted with a stopper, s, the tube D being, like C, connected with the chamber. D is connected by indiarubber tubing with the vertical tube I, which communicates freely with the very narrow tube H, the top of which is connected with the vessel G, and the bottom of which, curved up a little, dips into a vessel V full of mercury. There is an overflow from V to a trough T, as represented; and there is a clamp, c, by means of which communication between *D* and *I* can be established or broken. The vessel *G* is provided with a stop-cock, *g*.

The order of operations is as follows. The tubes being all completely occupied by air, fix the clamp *c*, remove the stopper *s*, lower the system of tubes *D*, *C*, and pour mercury into *F* and through its tube into *B*, until this mercury completely fills the tubes *C*, *D* and the chamber *A*. Close *A* with the stopper *s*, and raise the system *C*, *D* to its original position; open the clamp *c*, and let the mercury run from *B*, *C*, *D* into *I* up to the top, expelling air from *H* through the mercury in *V*. The mercury coming over at *a* into *H* will occupy a certain portion of *H*. Now let the stop-cock, *g*, be turned so that *G* is connected with the upper part of *H*. Mercury may be poured into *F* to keep up the flow from *a* through *H*, and the rate of supply of this mercury can be regulated by turning the stop-cock *f* more or less.

Now as each drop of mercury falls down through *H*, it forces the air in front of it down through the end of *H*; and hence the gas of *G* which keeps flowing into the upper part of *H* is perpetually driven down and out by the successive drops of mercury which fall over from *a*. If the mercury in *A* has fallen down through *D* into *I*, *H*, and *V*, the chamber *A* is a vacuum.

When the exhaustion of *G* has not been carried very far, the successive threads of mercury falling down *H* (and represented in the figure) succeed each other comparatively slowly, and they can be seen forcing the gas which reacts against their fall; but when the exhaustion is nearly complete, these drops fall much more freely through the now exhausted space; and when there is only a very small quantity of gas left in *G*, the drops falling from *a* on the top of the mercury surface, *H*, produce a sharp metallic sound, like that of a water hammer. This sound is an indication of a high degree of exhaustion.

When the exhaustion is complete the surface of the mercury in *H* will be at the barometric height above the level in *V*, and the difference of the level of the mercury in *B* and *C* will also be the barometric height.

The object of allowing the tube from *F* to dip into a much wider tube, *B*, is, partly, to let any air that may be carried down with the mercury from *F* escape into the atmosphere through the mercury in the wide tube, and partly to avoid filling *F* a very great number of times; this incessant filling will not be necessary if the tube *B* is very much wider than the other tubes.

The object of turning up the end of the tube *H* in the mercury in *V* is to allow the gas (whatever it may be) that is expelled from *G* through this end to be collected in another vessel, a tube from which is led to the point at the end of *H*.

The object of having the tubes *C*, *D* and the chamber *A* is (when *A* is vacant of mercury) to catch in this chamber any air that may have been carried over by the mercury from *F*, so that the exhaustion in the tube *H* shall be performed by mercury devoid of air. Hence the chamber *A* is called an *air trap*. So far as the principle of the Sprengel pump is concerned, we might dispense with the tubes *C*, *D* and the air trap, *A*, and connect *B* directly with *I*—and the Sprengel pump is, in fact, usually so represented.

38. Manometers. A manometer is an instrument for measuring the intensity of pressure of a condensed or an exhausted gas. The instrument has various forms, and the principle of all will be easily understood from that represented in Fig. 66.

Let *HFED* be a vertical bent glass tube, having a portion, *ED*, of one leg enlarged into a capacious reservoir, and let two necks *C*, *D*, project from this reservoir so that vessels may be connected with the reservoir by means of indiarubber tubes fitting on C and D. The leg HF is closed at the top.

Suppose the cross-section of this leg to be uniform, as also that of the reservoir except near its ends. Let mercury be poured into the instrument, and when the air thus imprisoned in HF assumes the temperature and pressure of the surrounding atmosphere (which enters at C and D) let AB be the level of the mercury in both legs, and let the number 1 be marked on the leg HF where the surface of the mercury stands, this point being, of course, in the prolongation of BA. The number indicates that when the air in HF occupies the length HI, at a given temperature, its intensity of pressure is 1 atmosphere. Let c be the length HI.

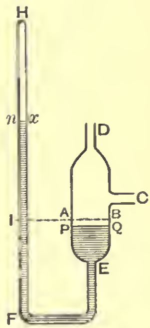


Fig. 66.

Now suppose that it is desired to fill a globe, or other vessel, with air at a great pressure and to measure the intensity of the pressure. Let the vessel be connected with the neck C, and let D be connected with a condensing pump. When this pump forces air in through D, and therefore through C into the vessel, the surface of the mercury falls in the reservoir ED and rises in the tube HF. After the pump has been working for any time, let n atmospheres be the intensity of pressure of the air in the reservoir (and therefore in the vessel which was to be filled); let PQ be the depressed surface in the reservoir, and let the surface of the mercury in HF be at x, which is x inches or millimetres above the level AB. If the intensity of pressure of the air in the reservoir is n atmospheres, the number n is to be marked at the level x on the tube HF. Let the depth of PQ below AB be y inches or millimetres; a = area of cross-section of HF, A = that of the reservoir; and let h inches or millimetres represent an intensity of pressure of 1 atmosphere. Then the difference of level between the point x and the surface PQ being $x+y$, and the volume of the air in HF being now $a(c-x)$ with an intensity of pressure denoted by $nh-x-y$, we have by Boyle's law

$$a(c-x)(nh-x-y)=ach.$$

But evidently $Ay=ax$; hence

$$a(c-x)\left\{nh-\left(1+\frac{a}{A}\right)x\right\}-ach=0,$$

which determines the number, n, to be marked on any part of the tube.

39. Hydrometers. These are instruments for the determination of the specific gravities of liquids and solids. We shall describe two only.

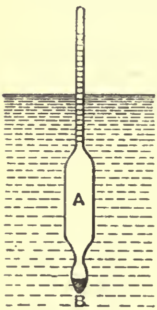


Fig. 67.

The Common Hydrometer, Fig. 67, is used for finding the specific gravities of liquids. It consists of a glass bulb, or cylinder, A, terminating at one end in a long narrow graduated stem, and at the other end in a small bulb, B, which contains a little mercury, the object of which is to keep the instrument vertical when it is immersed in a liquid.

If when immersed in one liquid it floats with a volume v immersed, and in another with a volume v', the specific weights of these liquids being w and w', respectively; and if W is the weight of the instrument itself, we have

$$v \,.\, w = W; \quad v' \,.\, w' = W,$$

$$\therefore \quad \frac{w}{w'}=\frac{v'}{v}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

so that the specific weights are inversely as the volumes immersed.

The volume of the portion AB irrespective of the stem can be found by graduating the stem (supposed of uniform bore) into any number of equal parts, and then observing the weight, W, of the instrument. Let masses p, p' be successively attached to the top of the stem, and with these let the instrument float in water up to the nth and n'th division, respectively. Then if V is the volume AB, and a the volume per division of the stem,

$$(V+na)\,w = W+p\,;\ (V+n'a) = W+p',$$

which determine V and a.

If when the hydrometer is immersed in two different liquids the readings on the stem are n and n', we have from (1)

$$\frac{w}{w'} = \frac{V+n'a}{V+na},$$

which shows that if a is very small, n and n' must be very widely different, i. e., the instrument is exceedingly sensitive to small differences of specific weight.

Sikes's Hydrometer is a form of the above in which the stem is a very thin flat strip of metal, for which, of course, a is very small.

Nicholson's Hydrometer. This hydrometer is employed to measure the specific gravities of both solids and liquids. It consists of a hollow metallic cylinder, A, Fig. 68, having a very fine stem on which there is a fixed mark, P; the lower end of the cylinder is connected by a wire with a closed cone, D, which is heavy enough to keep the stem vertical; the base, C, of this cone serves as a platform on which a solid body can be placed; the stem terminates in a cup, B, in which solids can be placed.

To find the specific gravity of a solid, place masses in B until the mark P is just sunk to the surface of the water; then place the given body in B: this will cause P to sink lower; remove weight from B until P again reaches the surface; if the weight removed is W, then W is the weight of the given body. Now remove the body from B to the platform C, and add weight, W', to B until P sinks to the surface; then W' is the weight of a volume of water equal to the volume of the body; and $\frac{W}{W'}$ is the required specific gravity.

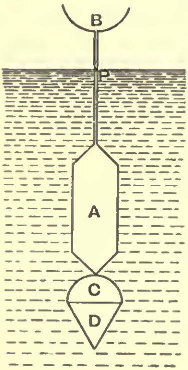


Fig. 68.

To find the specific gravity of a liquid, let H be the weight of the hydrometer itself; let the instrument be immersed in the given liquid; add weight, p, to B until P sinks to the surface; let p_0 be the weight which must be added to B to cause P to sink to the surface when the instrument is immersed in water; then evidently the specific gravity of the given liquid is $\frac{H+p}{H+p_0}$.

CHAPTER VII

STEADY MOTION UNDER THE ACTION OF GRAVITY

40. Steady Motion. When a fluid is in motion and we confine our attention to any point, P, in the space through which the fluid moves, it will be readily understood that the magnitude and direction of the velocity of the molecule which is passing through P at any instant may not be the same as the magnitude and direction of the velocity of the molecule which is passing through P at any other instant. If these should be the same at all instants, and if a like state of affairs prevails at all other points, the motion is said to be *steady*.

It is obvious, for example, that if a vessel is filled from a large reservoir of water, so that it is kept constantly full, while the liquid is allowed to flow out from an aperture made anywhere in the vessel, the motion at any fixed point in the vessel will be the same at all times.

41. Methods of Euler and Lagrange. It is at once obvious that the problem of the motion of a fluid acted upon by given forces may be attacked by two different methods. For, firstly, we may make it our aim to discover the condition of things—i. e., the magnitude and direction of the resultant velocity, and the pressure intensity—at each point, P, in space at all times, and thus, as it were, to obtain a map of the whole region—or a series of maps, if the motion is not steady—exhibiting the circumstances at each point as regards velocity and pressure.

Or, secondly, we may make it our aim to trace the path, and other circumstances, of each *individual molecule* throughout its whole motion.

The second object is much more difficult of attainment than the first, and, moreover, is not generally so desirable.

The first method is sometimes called the *statistical*, or the method of Euler; the second the *historical*, or the method of Lagrange.

42. Flow through a tube; work of gravity. Suppose a column of water to occupy at any instant a length AB of

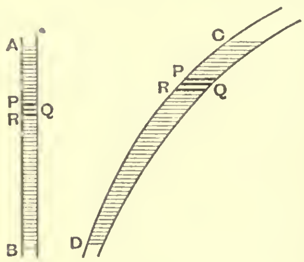


Fig. 69.

a straight vertical tube of uniform cross-section, and let the end B of the tube be open.

If in a small element, Δt, of time a mass, Δm, of water flows out, what is the work done by gravity on the water during this interval?

Divide the tube by a series of very close horizontal planes, $P, Q, R, \ldots$ into sections such that the mass included between each adjacent pair is Δm. If Δs is the distance between the middle points of successive layers, PQ, QR, &c., while Δm flows out the middle point of each layer will fall through the height Δs, and the work done by the weight of this layer will be

$$\Delta m \,.\, \Delta s \text{ or } g \Delta m \,.\, \Delta s,$$

according as we use gravitation or absolute measure of force. (If mass is measured in pounds and length in feet, the first expression gives the work in foot-pounds' weight, the second in foot-poundals.) If we use the first, the work is $\Sigma \Delta m \,.\, \Delta s$, which can be written in either of the forms

$$\Delta m\, (\Delta s + \Delta s' + \Delta s'' + \ldots),$$

$$(\Delta m + \Delta m' + \Delta m'' + \ldots)\, \Delta s,$$

in which, of course, the successive distances $\Delta s, \Delta s', \Delta s'', \ldots$ are all equal, and the successive weights $\Delta m, \Delta m', \Delta m'', \ldots$ are also all equal.

Hence the work is

$$\Delta m \times AB,$$

$$\text{or}\quad M \times \Delta s,$$

where M is the weight of the whole column.

The first expression shows that *the work done is the same as if the mass Δm which flows out at B fell through the height AB of the column.*

Precisely the same result holds if the shape of the tube is that represented in the right-hand figure. Let it be divided by close horizontal planes in the same manner as before. If now Δz is the *vertical* distance between the middle point of the layer, PQ, and the middle point of the next layer, QR, the work done in the descent of the first layer into the position of the second is $\Delta m \,.\, \Delta z$, so that the work done by gravity on the whole tube of liquid while the quantity Δm flows out at D is

$$\Delta m \times z$$

in gravitation units, where z is the difference of level of the ends C and D.

This is again the same as the work of carrying Δm from C to D.

43. Stream Lines. The actual path of a particle of a moving fluid is called a *stream line*. If at any point, A, Fig. 70, we describe a very small closed curve and at each point on the contour of this curve we draw the stream line, such as AP, and produce it indefinitely, we obtain a stream tube. When the fluid is a liquid, the mass contained between the normal sections of a tube at any two points, A, P, must always be the same; and therefore the same mass of fluid crosses every normal section of the tube per unit of time. Hence if v is the resultant velocity of the liquid at P and σ the area of the cross-section of the tube, the product

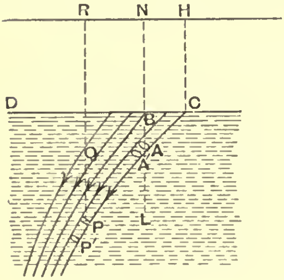


Fig. 70.

$$v \sigma$$

is constant all along the tube.

44. Theorem of Daniel Bernoulli. Consider at any instant the liquid contained in the stream tube between the normal sections at A and P, and suppose this liquid to occupy the volume $A'P'$ at the end of an infinitesimal element of time; let v_0, p_0, σ_0 be the velocity, pressure intensity, and cross-section of the tube at A; let v, p, σ be the same things at P; let z_0 and z be the depths of A and P below any fixed horizontal plane; let Δs_0 be the distance between the cross-sections at A and A', Δs being that between those at P and P'; and let $w =$ weight per unit volume of the liquid.

Now apply the equation of work and kinetic energy to the mass of liquid between A and P in the tube. The gain of kinetic energy in the small motion considered is

$$\text{kinetic energy of } A'P' - \text{kinetic energy of } AP,$$

in which, as the motion is steady, the kinetic energy of the portion $A'P$ is common to the two terms, and therefore disappears. Hence the gain of kinetic energy is

that of PP' – that of AA',

$$\text{or} \quad \Delta m \frac{v^2 - v_0^2}{2g}, \quad \dots \quad (1)$$

where Δm = weight of PP' = weight of AA'.

The external forces doing work on the column of liquid considered are

gravity, the pressure at A, the pressure at P.

The work of gravity is

$$\Delta m \,.\, (z - z_0), \quad \dots \quad (2)$$

by Art. 42.

The pressure at A is $p_0 \sigma_0$, and its work $= p_0 \sigma_0 \,.\, \Delta s_0$; the pressure at P is $p \sigma$, and its work $= -p\sigma \,.\, \Delta s$. Hence the work of the pressure is

$$-\frac{\Delta m}{w} \cdot (p - p_0), \quad \dots \quad (3)$$

since $\sigma \,.\, \Delta s = \sigma_0 \,.\, \Delta s_0$, i. e., the volume PP' = the volume $AA' = \dfrac{\Delta m}{w}$.

Equating (1) to the sum of (2) and (3), we have

$$\frac{v^2}{2g} + \frac{p}{w} - z = \frac{v_0^2}{2g} + \frac{p_0}{w} - z_0, \quad \dots \quad (4)$$

in other words, since A and P are any two points along the stream line,

$$\frac{v^2}{2g} + \frac{p}{w} - z = C \quad \dots \quad (5)$$

at every point of the stream line, C being a constant for the stream line chosen; but this constant may have different values as we pass from one stream line to another.

If, however, the liquid has a plane surface, such as DC (Fig. 70), at each point of which v is practically zero and p is constant, the constant C is the same for all stream lines. The equation (5) holds all along a stream line even if in its course the liquid flows along any number of *fixed* smooth surfaces; but if it meets surfaces which it sets in motion (as in a turbine) to the left-hand side must be added a term depending on the energy which it communicates to them at the point to which v belongs.

This result is the theorem of D. Bernoulli.

If at P we draw a vertical line, PQ, of such length that

$$p = w \cdot PQ,$$

the height PQ is called the *pressure head* at P. If also QR is drawn vertically of such length that

$$v^2 = 2g \cdot QR,$$

QR is called the *velocity head* at P. Let AB be the pressure head and BN the velocity head at A. Then (4) gives

$$PR = AN + z - z_0$$
$$= LN,$$

where $AL = z - z_0$ and is the perpendicular from A on the horizontal plane through P.

Since PL is horizontal, it follows that RN is horizontal. Hence the theorem of Bernoulli may be expressed in these words: *if at each point along a stream line there be drawn a vertical line whose length = the pressure head + the velocity head at the point, the extremities of all these vertical lines lie in the same horizontal plane.*

If the liquid has a horizontal surface, CD, at rest at all points of which the intensity of pressure is constant (e.g., that of the atmosphere), the extremities of these lines drawn at *all* points of the liquid, and not merely along the same stream line, will all lie in the same horizontal plane. If CH is the pressure head on the surface CD (about 34 feet if CD supports the atmospheric pressure), the extremities $R, \ldots$ of the vertical lines drawn at all points $P, \ldots$ lie in the horizontal plane through H.

For a liquid in equilibrium, Q coincides with R, since $QR = 0$, and it has already been shown that the extremities of all vertical lines representing pressure heads lie in the same horizontal plane. The theorem of Bernoulli is the generalization of this result for a liquid in steady motion.

An approximate method of indicating the value of p, the pressure intensity at any point P in a moving liquid consists in inserting a vertical glass tube, open at both ends, into the liquid, one extremity of the tube being placed at P. The liquid will rise to a certain height in this tube and remain at rest. Thus, if the tube is so long that the upper end is above the free surface CD, the liquid would rise in it to the height PQ, the remainder of the tube being occupied by air. Such a tube is called a *pressure gauge*; but it is evident that it does not strictly measure the pressure, since the glass must, to some extent, alter the motion of the liquid.

45. Flow through a small orifice. Let Fig. 71 represent a vessel containing a liquid whose level is LM which flows out through a small aperture made anywhere in the side of the vessel, and let the thickness of the side be so small that the liquid touches the inner edge, AB, of the orifice and thence passes out without touching the outer edge or any intervening part of the aperture.

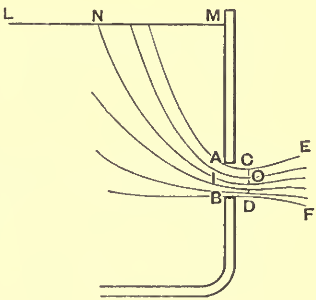


Fig. 71.

The curved lines in the figure represent the stream lines, or paths of particles, the forms and positions of which cannot, however, be determined mathematically.

We are obliged to have recourse to experiment for certain facts concerning the issuing jet. Firstly, it is found that after leaving the orifice AB, this jet contracts to a minimum cross-section, CD, beyond which, of course, the jet widens out again. This minimum cross-section is called the *vena contracta.*

The ratio of the area of the vena contracta to that of the orifice AB is called the *coefficient of contraction.*

For a circular orifice whose diameter is AB, if CD is the diameter of the vena contracta, it has been found experimentally that

$$\frac{CD}{AB} = \cdot 79,$$

so that if S is the area of the orifice and σ that of the vena contracta,

$$\frac{\sigma}{S} = \cdot 624.$$

It has also been found that the distance, IO, between the orifice and the vein is somewhere between $\cdot 39 \times AB$ and $\cdot 5 \times AB$, where, as before, AB is the diameter of the orifice; the uncertainty arising from the fact that in the neighbourhood of the minimum section the diameter of the jet varies very little. All the streams which pass through the vena contracta cut its plane perpendicularly. By consideration of the general equations of motion, it will follow from this fact that the intensity of pressure is the same at all points in the vena contracta. At all points on the outer surface, ACE, BDF, of the jet the pressure intensity is, of course, the same as that of the atmosphere, if the jet flows into the atmosphere; also the velocities at all points of the vein are equal; but in the interior of the jet this pressure intensity does not exist, except at points in the plane of the vena contracta.

Of course at the orifice AB the directions of motion are not all perpendicular to the cross-section of the jet, neither are the velocities all the same at points in this section.

46. Theorem of Torricelli. In the case of a jet escaping into the air, the velocities of particles in the vena contracta are expressed by a very simple formula.

In (4) of Art. 44, let p and z refer to a point, O, in the vena contracta while p_0, z_0 refer to the point, N, of the stream line through O which is on the free surface of the liquid. Then $p = p_0$, as we have said above, and as the velocities at the surface LNM are all very small, we may consider $v_0 = 0$.

Hence

$$\begin{aligned} v^2 &= 2g(z - z_0) \\ &= 2gh, \quad \quad (a) \end{aligned}$$

where h, or $z - z_0$, is the vertical depth of the vena contracta below the free surface LNM.

Hence when the particles reach the vena contracta, they have the same velocity as if they fell directly from the free surface. This is known as Torricelli's Theorem. Obviously it holds with considerable exactness in the case of a *small* orifice only.

If a liquid devoid of friction escapes from a small orifice in a vessel in which the free surface is maintained at a constant level, the velocity in the vena contracta is theoretically given by the equation (a).

In the case of water, however, it is found that the velocity is not quite equal to this amount, but is very nearly a constant fraction, μ, of the value given by (a). The fraction μ is nearly equal to unity (about ·97). We may therefore put

$$v = \mu\sqrt{2gh}.$$

Again, if S is the area of the aperture, and c denotes the coefficient of contraction (Art. 45), the area of the cross-section of the vena contracta is cS; so that the volume of water issuing from the vessel per unit of time is

$$c\mu S\sqrt{2gh}.$$

If the unit of length is a foot and the unit of time a second, this is the discharge in cubic feet per second, and multiplying it by w, the mass per unit volume (in the case of water $62\frac{1}{2}$ lb.), we obtain the mass discharged per second.

The product $c\mu$ may be taken as $\cdot 62$.

To find the time in which a vessel of any form, ACB, filled originally to a level AB, Fig. 72, with water will be emptied through a small orifice at a point V, let PQ be the level at any time t, let z be the vertical distance between PQ and AB, let x be the depth of V below PQ, $S =$ area of orifice, and $A =$ area of the section PQ of the vessel.

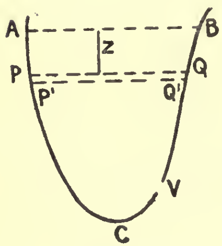


Fig. 72.

Then in the time dt the volume of the liquid discharged is that between PQ and $P'Q'$, i.e., Adz; but in this time the volume discharged at V is

$$\cdot 62\, S\sqrt{2gx}\,.\,dt,$$

if we can regard the velocity at each point of PQ as practically zero. Also $dz = -dx$, so that

$$\cdot 62\, S\sqrt{2gx}\,.\,dt = -Adx.$$

Now A is known in terms of x from the shape of the vessel; hence if the depth of V below AB is h,

$$t = \frac{1}{\cdot 62\, S\sqrt{2g}} \cdot \int_0^h \frac{Adx}{\sqrt{x}}\,.$$

EXAMPLES.

1. A cylindrical barrel, the area of whose cross-section is 12 square feet, and whose axis is vertical, is filled to a height of 4 feet with water; in what time will it be emptied through a hole whose area is half a square inch placed in the bottom of the barrel? (Take $g = 32$.)

Result. $46^{m}\ 27^{s}$.

2. In what time will a cylinder of radius a feet be emptied through a hole of radius r inches in the bottom, if the cylinder was filled to a height h feet?

Result. $\frac{1800}{31} \cdot \frac{a^2}{r^2} h^{\frac{1}{2}}$ seconds.

3. Show that if the level of the water had been kept for this time at its original height, twice as much water would have been discharged.

4. Find the time in which a vertical hollow cone of volume V filled with water to a height h will be emptied through a small orifice of area σ at its vertex.

Result. $\frac{6V}{3 \cdot 1 \times \sigma \sqrt{2gh}}$ seconds.

5. If the level had been kept constant for this time, how much water would have been discharged?

Result. $\frac{6}{5} V$.

6. Find the time taken by a hollow sphere of radius r filled with water to empty itself through a small orifice at the lowest point.

Result. If $V =$ volume of sphere, $\sigma =$ area of orifice,

$$\frac{4V}{3 \cdot 1 \times \sigma \sqrt{2gr}}.$$

47. Flow through a large orifice. The determination of the discharge through a large orifice cannot be satisfactorily accomplished by theory.

Suppose, for example, that the orifice is a rectangle, $ABCD$, with vertical and horizontal sides, and that LM (Fig. 73) represents the level of the free surface in the vessel, the flow being supposed to take place through the orifice towards us as we look at the figure.

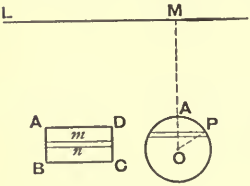


Fig. 73.

Divide the area of the aperture into an indefinitely great number of narrow horizontal strips, of which that between the horizontal lines m and n is the type.

Let the depths below LM of the lines AD and BC be h_1 and h_2, respectively, those of the lines m and n being z and $z+dz$. Let $AD = b$; then, supposing that the aperture between m and n alone existed, the volume of the discharge would be given by (3) of last Article, in which $S = bdz$. Denoting the product $c\mu$ by k, and by dQ the mass discharged per unit time through the strip, we have

$$dQ = kbw\sqrt{2gz}\,.\,dz. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now the assumption that k is constant for all the strips enables us to find Q, the total discharge; but clearly this assumption cannot be strictly correct, for each strip does not discharge as if it alone existed as an aperture.

Assuming k to be constant, we integrate (1) from $z = h_1$ to $z = h_2$, and obtain

$$Q = \tfrac{2}{3}\,kbw\sqrt{2g}\,(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}}). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To calculate the energy per second which flows through the orifice, if v is the velocity of the portion dQ, its kinetic energy is

$$\frac{v^2}{2g}\,.\,dQ, \text{ i.e. } zdQ.$$

Energy per unit time is called *Power*. Hence if dP is the power of this flow,

$$dP = kbw\sqrt{2g}\,.\,z^{\frac{3}{2}}\,dz,$$

$$\therefore \quad P = \tfrac{2}{5}kbw\sqrt{2g}\,(h_2^{\frac{5}{2}} - h_1^{\frac{5}{2}}). \quad . \quad . \quad . \quad (3)$$

If in this expression length is measured in feet, time in seconds, and w in pounds, since the unit of power called a Horse-Power is 550 foot-pounds' weight per second, we get the Horse-Power of the discharge equal to the right-hand side of (3) divided by 550.

If AB is small compared with the depth h_1, and if h is the depth of the centre of area of the orifice, we can easily find from (2) that

$$Q = kwS\sqrt{2gh},\quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

where $S =$ the area of the orifice. For if $AB = 2a$, we have $h_2 = h\left(1 + \frac{a}{h}\right)$, $h_1 = h\left(1 - \frac{a}{h}\right)$, and expanding in powers of $\frac{a}{h}$, we see that the term $\frac{a^2}{h^2}$ disappears, and (a) is true if we neglect the small fraction $\frac{a^3}{h^3}$.

As another example of the same kind, suppose the orifice to be circular (Fig. 73), and that powers of $\frac{r}{h}$ beyond the second are negligible. Then it will be found that

$$Q = \pi kr^2 w\sqrt{2gh}\left(1 - \frac{r^2}{32h^2}\right),$$

where h is the depth of the centre O.

Examples.

1. *The Syphon.* We now take a few common practical illustrations of the application of equation (4), Art. 44, which applies to the motion of a liquid acted upon by gravity. The

first of the simple examples is furnished by the common syphon which is employed for the purpose of raising a liquid out of a vessel and lowering it into another vessel. The operation might, of course, in many cases be directly performed by taking the first vessel in the hand and pouring out the liquid bodily into the second; but if, as often happens, the liquid is Sulphuric or Nitric Acid, which it would be most undesirable to pour out with splashing, this method would not answer, and a syphon is used. The syphon is a bent tube (usually of glass) open at both ends, and with unequal branches.

Suppose M (Fig. 74) to be the vessel which it is desired to empty into another (not represented in the figure), and suppose the liquid to be water.

A bent tube, $DABC$, (the syphon) whose branch BC is longer than the branch BD is first filled with water, and the apertures at D and C held closely by the fingers. The end D is then inserted into the liquid in the vessel M, the fingers removed from D and C, and the tube held in the hand. The result will be a flow of the liquid through C until, if D is kept close to the bottom of the vessel, nearly all the liquid is removed.

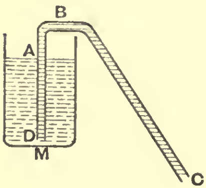


Fig. 74.

Let p_0 be the atmospheric intensity of pressure, which exists on the surface of the liquid at A and also at C; v_0, the velocity of liquid on the surface A, is nearly zero; if $v =$ velocity of efflux at C, $z =$ depth of C below A, equation (4) of Art. 44 gives, since $ADBC$ may be taken as a simple stream line,

$$\frac{v^2}{2g}+\frac{p_0}{w}-z=\frac{p_0}{w},$$

$$\therefore \quad v=\sqrt{2gz},$$

so that the flow will continue all through if C is at a lower level than D. Of course there will be a small residue of liquid in M, because when nearly all has flowed out, air will enter the syphon at D.

If the liquid to be removed is an acid, as sulphuric or nitric, the syphon must be filled with it at the beginning by first inserting the end D into the vessel and then sucking the air out through C until the liquid rises in the syphon and falls in the leg BC to a lower level than A; and this suction may be effected by joining another tube to the end C by means of a short piece of indiarubber tubing which can be subsequently removed.

2. *Hero's Fountain.* Hero of Alexandria (120 B.C.) constructed a fountain, which is represented in Fig. 75. It consists of two glass globes, M and N, and a dish, DD. Each globe is partly filled with water and partly with air at the atmospheric pressure. The globes are fitted with necks and are held together by two glass tubes, A, B, each open at both ends, which pass through necks fitted to the globes. The extremities of A are in the air in the globes; the lower extremity of B dips nearly to the bottom of the liquid in M, while its upper end barely projects into the dish DD. A third tube, C, open at both ends, passes through the neck of N, its lower end dipping nearly to the bottom of the liquid in N, while its upper end projects beyond the upper surface of the dish.

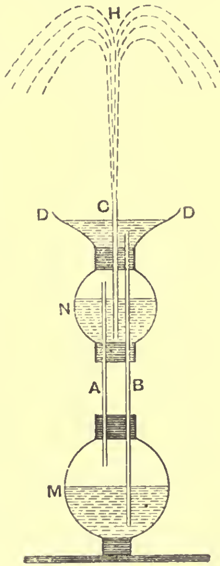


Fig. 75.

In this state of affairs the water is at rest in both vessels, the intensity of pressure on both water surfaces being p_0, that of the atmosphere.

If now water is poured into the dish, it will fall through B into M and drive some of the air into N where the surface pressure on the water becomes greater than p_0, and as a result the water from N is forced up through the tube C into the air.

To calculate the height to which it rises, let z be the difference of level between the water in M and that in N; let c = difference of level between that in D and that in N; and let h = the height of the top of the jet above the water in D.

Then, since the pressure intensity of the air in N = that in $M = w(z+c)+p_0$; since the velocity of the water at the surface in N is nearly zero, and is also zero at H, where the pressure intensity is p_0, we have from (4) of Art. 44

$$\frac{p_0}{w} + z + c - (c + h) = \frac{p_0}{w},$$

$$\therefore \quad h = z,$$

i.e., the height of the jet above the water in D is equal to the difference of level in M and N.

3. *Mariotte's Bottle.* It is sometimes desired to produce a narrow jet of water flowing for a considerable time with constant velocity. Of course a very large reservoir with a very small aperture made in the side would produce the result; but such a reservoir is not always at hand. The result can also be produced by means of a broad flask fitted with a stop-cock near the bottom. Fig. 76 represents the flask. The stop-cock (not represented) is fitted at C, and the aperture is supposed to be very small compared with the cross-section of the flask. The flask is first quite filled with water, the stop-cock being closed. In the top of the flask there is a neck fitted with a cork, and into this is inserted a tube, HD, open at both ends, the tube also being quite filled with water.

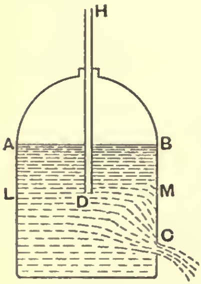


Fig. 76.

Now let the stop-cock be opened, and water will flow out, because the atmosphere presses at H and at the outside of C, and between C and H there is a column of water. The water that first flows out comes from the tube HD alone, the flask remaining filled to its upper surface; and, moreover, the velocity of efflux will be variable as the level sinks in HD. But when the tube is emptied of water, some air will be forced through D by superior atmospheric pressure, and it will rise to the upper part of the flask, and will begin to force down the water of the flask.

This being the case, the intensity of pressure at D in the water is p_0, the atmospheric intensity, and we may assume that p_0 is also the intensity all over the horizontal plane, LM, through D, because the motion of particles in this plane is very slow. The air at the top aided by the water above LM will keep the pressure intensity approximately equal to p_0 at points in LM other than D.

Now let $z = CM =$ vertical distance of orifice below D, the lower extremity of the tube; let v_0, p_0 in (4) of Art. 86 refer to D, while v is the velocity at C. Then, since p_0 is also the pressure intensity at C,

$$\frac{v^2}{2g} + \frac{p_0}{w} - z = \frac{p_0}{w},$$

$$\therefore \quad v = \sqrt{2gz},$$

which shows that the velocity is constant whatever the position of the upper surface, AB, of the water in the flask.

The tube must, of course, have such a position that D is above the aperture. If the water, instead of escaping into the atmosphere, escapes into a medium (gaseous or liquid) in which the pressure intensity at C is p, we shall have

$$v^2 = 2g\left(z + \frac{p_0 - p}{w}\right).$$

This vessel is known as Mariotte's Bottle.

4. *Thomson's Jet Pump.* If water flows with steady motion through a horizontal pipe of variable cross-section (Fig. 77), the

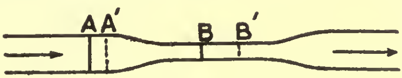


Fig. 77.

pressure intensity at a narrow part of the pipe is less than it is at a broad part. Let A be a part at which the section is broad and B a part at which it is narrow; let v and p be velocity and pressure intensity at A, and v', p' those at B; let S and S' be the cross-sections at A and B; in a small interval of time, Δt, let the water at A come to A' and that at B come to B'. Employing the equation of work and energy to the mass of water in the pipe between A and B in occupying the space between A' and B', the work done on it is $pS\Delta x - p'S'\Delta x'$, where Δx is the distance between the sections at A and A', and $\Delta x'$ the distance between those at B and B'. Also, as at p. 155, the gain of kinetic energy is $wS'\Delta x' . \frac{v'^2}{2g} - wS\Delta x . \frac{v^2}{2g}$.

Hence

$$pS\Delta x - p'S'\Delta x' = wS'\Delta x' \cdot \frac{v'^2}{2g} - wS\Delta x \cdot \frac{v^2}{2g}.$$

Now, since the liquid is incompressible, $S'\Delta x' = S\Delta x$; hence

$$\frac{v'^2}{2g} + \frac{p'}{w} = \frac{v^2}{2g} + \frac{p}{w}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

But $\Delta x = v\Delta t$, and $\Delta x' = v'\Delta t$, $\therefore \; Sv = S'v'$, $\therefore \; v' > v$, and (a) shows that $p' < p$.

On this principle is founded Professor James Thomson's jet pump which can be employed for draining marshy land.

ABC (Fig. 78) is a horizontal pipe with a narrow cross-section at *B*, into which is fitted a vertical pipe, *D*, which is plunged into the marshy soil. This pipe *D* ends inside the pipe *ABC* in a nozzle situated near *B*. A steady flow of water from a reservoir enters at *A* under atmospheric, or nearly atmospheric, pressure. At *B* the pressure is much less; and if *D* dips into water at atmospheric pressure, this water will rise and, passing through the nozzle (if the pipe *D* is not very long), will be carried through *B* and discharged through the mouth *C* wherever desired.

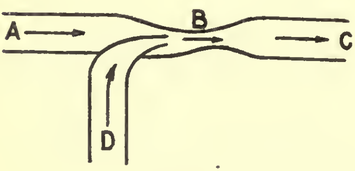


Fig. 78.

The difference between the water pressures at *A* and *B* (Fig. 78) can be exhibited by inserting vertical glass tubes into the pipe at *A* and *B*. The water will then ascend from the pipe into these tubes and will stand at a higher level in the tube at *A* than in the tube at *B*. If the water in the tube inserted at *A* reaches to a point *H* in the tube and the water in the tube at B reaches to H', and if we prolong AH to K so that $HK = \frac{v^2}{2g}$, v being the velocity at A, and similarly prolong BH' to K' so that $H'K' = \frac{v'^2}{2g}$, v' being the velocity at B, the points K and K' will stand at the same level, which will be that of the water in the reservoir from which the pipe AB is supplied. This follows from equation (a).

48. Theory of Turbines. In connexion with the motion of water along surfaces which are themselves in motion, the following result in elementary dynamics is useful: *if a particle moves under the influence of force from one position to another, the change in its moment of momentum about any fixed axis is equal to the time-integral of the moment of the force about the axis from the one position to the other.*

Let a particle of mass m (pounds, suppose) at the point P, Fig. 79, have at the time t a velocity v represented by PQ; at the end of an extremely small interval dt let its velocity be represented by the vector PR; then the vector QR, or PS which is equal and parallel to QR, is $a \,.\, dt$, where a denotes the resultant acceleration of the particle at P. Then by Newton's Second Axiom PS is the direction of the resultant force acting on the particle, and this force, in gravitation units (pounds' weight) is $\frac{m \,.\, a}{g}$.

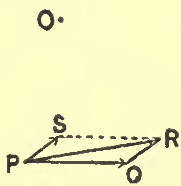


Fig. 79.

Now by the principle of moments, since the vector PR is the resultant of PQ and PS, the moment of PR about any axis, represented by O, is equal to the sum of the moments of PQ and PS about that axis;

$$\therefore \text{ moment of } PR - \text{moment of } PQ = \text{moment of } PS.$$

Now $m \times PR$ and $m \times PQ$ are the momenta of m at the beginning and end of the interval dt; therefore the change of moment of momentum = moment of $madt$ = moment of $g \cdot Pdt$, if P is the resultant force acting on the particle.

Hence if the initial and final moments of momentum of the particle are M_2 and M_1, and L denotes at any instant the moment of the resultant force about the axis, we have

$$M_2 - M_1 = \int Ldt,$$

the integral being taken throughout the whole time interval.

Let AB, Fig. 80, be a smooth plane curve rotating with angular velocity ω about an axis at O perpendicular to its plane, and let a particle P be moving along the rotating curve. Then if N is the reaction of the curve at P, acting along PN, and if at the end of the time dt the curve comes to $A'B'$ and the particle to P', the work which the particle does on the curve may be calculated by supposing the particle not to move along the curve, but to be carried with it to Q, such that $PQ = \omega r dt$, and to move along the curve, considered fixed, from Q to P'. In this latter displacement N does no work, and in the former N does on the curve the work $N . \omega r dt . \sin \phi$, where $\phi = OPN$, i.e., $\omega . N \varpi dt$, where ϖ is the perpendicular from O or N. The work which N does on the particle is, of course, the same with a negative sign.

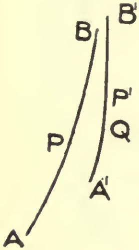


Fig. 80.

Let AB (Fig. 81) be a vane forming a part of a wheel which is rotating about a fixed axis O, so that u is the velocity of the end A, this velocity being perpendicular to OA; and suppose a particle of water moving with absolute velocity V on reaching A to enter along the vane without shock on the vane. Then if AT is the direction of the tangent at A to the vane, this must be the direction of the *relative* velocity of the water and the vane at A; let AC represent u and AD represent V; then applying u reversed to the particle of water and combining V with u reversed, we obtain the relative velocity. Draw AC' equal and opposite to AC; then the resultant of AD and AC' must be along AT. If α is the angle TAC and θ the angle DAC,

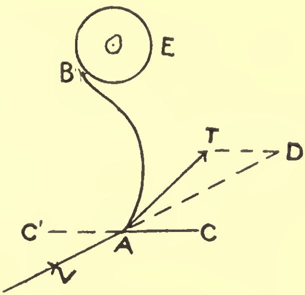


Fig. 81.

$$V \sin(\alpha - \theta) = u \sin \alpha.$$

Since there is no relative velocity normal to the vane, there will be no sudden impulse or blow received by the water on the vane, and therefore no sudden loss of kinetic energy.

The velocity V of the water can be resolved into a tangential component, V_t, along AC and a radial component, V_r, along AO.

Now suppose a channel A to be carried by a wheel D (Fig. 82) which rotates round O with angular velocity ω; suppose a continuous supply of water from the outside to be given to the wheel and the water to be discharged from the channel at its other end into a space OC surrounding the axis and to flow away continuously out of the machine in a direction perpendicular to the plane of the figure. Further, suppose the angular velocity of the wheel to be always the same—by making the wheel do some work, against external resistance, at a constant rate. Then the state of affairs inside the channel will be the same at all times, so that, for example, the moment of momentum about O of the water which fills the channel will always be the same.

Imagine the channel to be divided into a very great number of sections 1, 2, 3, 4, ... all of the same volume. Let this channel revolve into the position A' in the time dt. We have represented a particle m of water just outside A which is ready to enter the channel, and which actually enters it when the channel reaches the position A'. This particle has the velocity requisite for avoiding shock, i. e., it has tangential velocity V_t and radial V_r, as above.

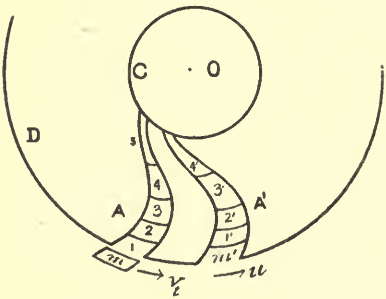


Fig. 82.

Now let the construction of the vanes be such that the water ejected into the space OC has its absolute velocity at exit directed towards O; and consider the change of moment of momentum, about O, of the water in A. The section 1 moves into the compartment 1′ in A'; the section 2 moves to 2′, and so on; the last section (5 in the figure) is ejected radially and has no moment of momentum about O.

Now let M be the moment of momentum of the water 1, 2, 3, 4, ... ; also M is the moment of momentum of m', 1′, 2′, 3′, 4′, ... ; and the new moment of momentum of the water 1, 2, 3, 4, ... is that of 1′, 2′, 3′, 4′, ... which is M minus the moment of momentum of m'. If dw is the weight of m, the moment of momentum of m' is $dw.aV_t$, where a is the outer radius of the wheel, i. e., the distance of m' from O. Hence the change produced in time dt in the moment of momentum of the water in A is

$$M - dw\,.\,aV_t - M, \quad \text{or} \quad -dw\,.\,aV_t\,;$$

and this is, as above proved, equal to dt multiplied by the moment about O of the normal reactions of the vane on the elements of water 1, 2, 3, 4, ...

If N is the normal reaction per unit length of the vane at any point P of the channel A, the total moment of force is

$$-\int N\varpi\, ds,$$

the integral being taken along the whole length of the channel A. Hence

$$\frac{dw\,.\,aV_t}{g} = dt \int N\varpi\, ds.$$

If we denote $\frac{dw}{dt}$ by W, the weight of water entering the channel per unit time,

$$\frac{WaV_t}{g} = \int N\varpi\, ds.$$

But we have proved above that the work, dE, done in time dt on the vane is $\omega\, dt \int N\varpi\, ds$; hence

$$\frac{WaV_t}{g} = \frac{1}{\omega}\frac{dE}{dt},$$

and since ωa is u, the circumferential velocity of the wheel at its outer rim, we have

$$\frac{WuV_t}{g} = \frac{dE}{dt},$$

which gives the energy communicated per unit time by the water to the wheel.

If the wheel has a large number of channels such as A, then W is the total weight of water entering the chambers per unit of time. If, as is often the case, the direction of the tangent AT (Fig. 81) to the vane is radial, $V_t = u$, and the energy per second given to the wheel is $\frac{Wu^2}{g}$.

The case just described is that of the *Reaction Turbine*, so called because it is worked by the continuous pressure of

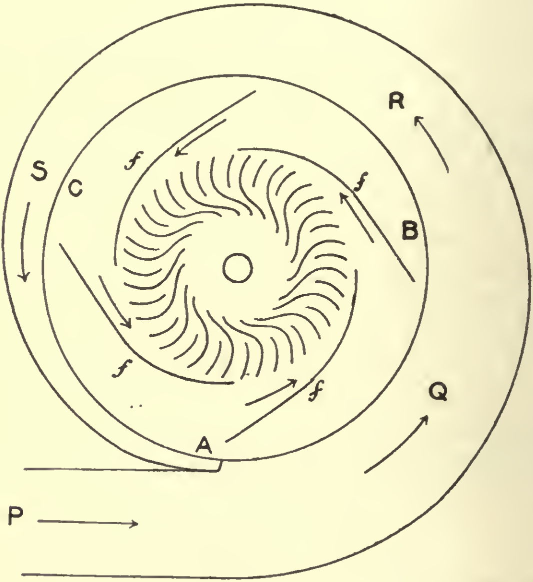


Fig. 83.

water flowing along curved blades, forming curved spokes of a revolving wheel. When the water enters the wheel at its outer rim and is ejected through the hollow axis of the wheel the machine is called an *inward flow turbine*.

Fig. 83 represents the main features of the inward flow reaction turbine invented by Professor James Thomson.

Water enters a large pipe at P, flows round a chamber QRS, which surrounds another fixed chamber ABC. In this latter are fixed certain curved blades, f, f, f, f, called *guide blades*, the chamber ABC in which they are fixed being called the *guide chamber*. These fixed blades guide the water to apertures on the outer rim of a wheel which can rotate round a fixed central axis, and the spokes of this wheel are a large number of curved blades or vanes such as that already described in Fig. 81. The water from the guide chamber ABC enters the wheel, of course, at apertures all over its rim, although the ends of the guide blades reach the rim of the wheel at only four of the apertures in the rim. We can assume that the whole of the water flowing into the rim at the left-hand side of any guide blade f strikes the spoke vanes of the wheel at an angle not greatly different from that at which the water at the end of the blade f strikes the wheel there. Although the guide blades f have been said to be fixed, they are really capable of some adjustment, each blade f having a pivot fixed in it near the end which is close to the wheel, and the rotation of these blades is easily effected from the outside of the machine.

Now in the equation p. 171, since a is constructed once for all and cannot be varied, we see that if u is to be constant—that is, if the wheel is always to be run at constant speed—while from any cause V varies, θ must be capable of being altered; and this is effected by slightly rotating the guide blades f round their pivots.

The chamber QRS, from which the water enters the guide chamber, is called the *vortex chamber*. The space between these two chambers narrows very much up to a point A close to the supply pipe P; and here the water enters the guide chamber through an aperture at A, so that the velocity with which it enters this chamber is greatly increased.

Fig. 83 represents a section, perpendicular to the axis of the revolving wheel, of the wheel itself and its surrounding chambers. The whole is, of course, boxed up in a casing through which the axis of the wheel projects, so that none of the interior of the machine (shown in the figure) is visible from the outside.

To the revolving axis of the wheel is fixed on the outside a pulley, round which is passed a belt which can be used to drive any revolving machine, such as a dynamo, which is to be set in motion by means of the turbine.

The *efficiency* of the turbine is measured by the ratio of the useful work which it does in any time to the energy of the water which enters it in that time. Assuming that there is no loss by friction, the energy imparted to the wheel by a weight W of water is $W\frac{u\,.\,V_t}{g}$ (p. 173), where V_t and u are the absolute circumferential velocities of the water at entrance to the wheel and of the wheel itself. If the total head of the water before entering the machine is h, the possible available energy per second is $W\,.\,h$, and the efficiency is $\frac{uV_t}{gh}$.

If W pounds of water are supplied per minute to the turbine, V_t and u being in feet per second, the horse-power developed by the machine is

$$\frac{WuV_t}{33000g},$$

where $g = 32$, approximately.

We see the reason for arranging the blades so that the absolute velocity of the water at exit from the wheel is radial; for if the terminal section $5'$ (Fig. 82) had moment of momentum about O, the change of moment of momentum of the water in the channel A would not be quite so great as $dw \, . \, aV_t$, and the energy given to the wheel would be diminished.

If the height of the supply water above the turbine is h, and p_0 is the pressure (atmospheric) on its surface, we have by Bernoulli's theorem

$$\frac{p_0}{w} = \frac{V^2}{2g} + \frac{p}{w} - h,$$

where V and p are the velocity and pressure at entrance to the wheel, and if E is the work which the water does per unit weight on the wheel and V_r' is the velocity of the water at exit, each side of the above equation is equal to

$$\frac{V_r'^2}{2g} + \frac{p_0}{w} - h + E,$$

$$\therefore \quad E = h - \frac{V_r'^2}{2g}.$$

If we neglect V_r', we get

$$\frac{uV_t}{g} = h,$$

an equation which is only approximately true.

If the blades are radial at the outer circumference of the wheel, $u = V_t$, and then we have

$$V_t^2 = gh,$$

which shows that the circumferential velocity must be that due to half the total head of water.

There is a simple form of turbine, sometimes used when only a small power is required, in which the guide chamber is placed above the wheel chamber, and the water enters through the common axis of these chambers. This is called an *axial flow turbine.*

In all forms of turbines there are two conditions to be satisfied for high efficiency, namely—

1. The water must enter the wheel without normal velocity on the blades, because normal velocity entails shock, and this entails sudden loss of energy;

2. The water must leave the wheel with an absolute velocity which is wholly radial.

Turbines are usually divided into two classes—*reaction turbines* and *impulse turbines.* These names are not accurately expressive. By reaction turbines are meant those in which the water when it reaches the wheel blades is not only in motion but also under some pressure exceeding the pressure of the atmosphere. If the water was originally derived from a source having a height H above the level at which it enters the machine, its total head at entrance

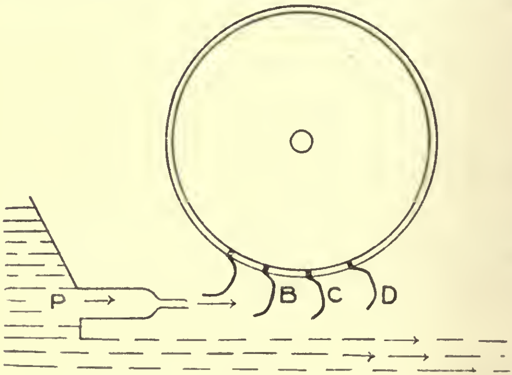


Fig. 84.

would be $h_0 + H$, where h_0 is the head due to the atmosphere. When it leaves the wheel chamber its head is h_0, and when it is driving the wheel it has a head intermediate to $h_0 + H$ and h_0; in addition to this it has its kinetic energy when driving the wheel. If, however, the turbine is such that when the water begins to drive the wheel its head is simply h_0, i. e., if the excess head, H, is wholly turned into kinetic energy before striking the blades of the wheel, the machine is called an *impulse turbine.* An example of this latter kind is the Pelton Wheel, represented roughly in Fig. 84.

To the rim of the wheel are attached curved buckets, B, C, D, ... all round, only four of which are shown.

The water from a reservoir or from a stream in which a dam is fixed issues through a sluice or through a pipe P fitted with a nozzle, and impinges with considerable velocity on the bucket B, running up into the interior of the bucket. As the wheel revolves, the water falls out of the bucket, and the bucket has a lip at its lower end over which all the water is emptied shortly after the bucket has left its lowest position, as at D. The buckets are not spherical cups; their interior surfaces are more nearly cylindrical, and each cup is divided by a partition into two compartments separated by a diaphragm represented in Fig. 85 by fg, which is in the direction of the radius of the wheel.

Now, reverting to the case of a particle projected along a smooth curve (Fig. 80) which is moving in its own plane, let v_x and v_y be the components of the absolute velocity of the particle at P, while α and β are the components of the absolute velocity of the point I of the curve itself, parallel to two fixed axes of x and y. Then if θ is the angle made with the axis of x by the tangent at P, and w is the weight of the moving particle, the equations of motion of the particle are

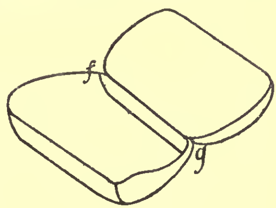


Fig. 85.

$$\frac{w\dfrac{dv_x}{dt}}{g} = -N\sin\theta, \quad \frac{w\dfrac{dv_y}{dt}}{g} = N\cos\theta,$$

$$\therefore \quad \cos\theta\frac{d}{dt}(v_x) + \sin\theta\frac{d}{dt}(v_y) = 0.$$

Now the components of the relative velocity of the particle and tube at P are $v_x - a$ and $v_y - \beta$; and since the resultant relative velocity is along the tube

$$\frac{v_y - \beta}{v_x - a} = \tan\theta.$$

Hence the above equation becomes

$$(v_x - a)\frac{d}{dt}(v_x) + (v_y - \beta)\frac{d}{dt}(v_y) = 0.$$

So that if a and β do not vary with the time, we have

$$(v_x - a)^2 + (v_y - \beta)^2 = k^2 = \text{Constant},$$

or, in other words, the relative velocity is constant in magnitude.

Now in the case of the Pelton wheel the water entering the bucket strikes the partition fg, separating the two halves of the bucket, and is directed sideways, as represented in Fig. 86, which is a section of the bucket perpendicular to fg.

Each such section of the bucket can be considered as a smooth blade moving with a velocity u in the direction of the water jet, and if V is the velocity of the jet, $V - u$ is the relative velocity of the water and bucket both at entrance and exit. The absolute velocity of the water at exit in the sense of the jet is $u-(V-u)$, i.e., $2u - V$; so that if $u = \frac{1}{2}V$, the water will at exit drop out of the bucket

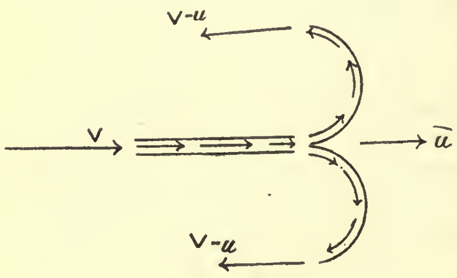


Fig. 86.

with no velocity, and the whole of its original kinetic energy, $W\frac{V^2}{2g}$, will be abstracted from it by the wheel.

If, then, the wheel is made (by doing work in lifting, or in driving machinery) to run with a circumferential velocity equal to half that of the impinging water, the efficiency would be perfect, on the supposition that no friction is encountered in the machine.

The efficiency of the wheel for any value of u is thus found: if W is the weight of water supplied to the wheel in any time, the loss of energy of the water is

$$\frac{WV^2}{2g} - \frac{W(2u-V)^2}{2g}, \text{ or } \frac{4Wu(V-u)}{2g};$$

and if this is wholly given to the wheel, the ratio of the kinetic energy of the wheel to that of the water is the ratio of this to

$$\frac{WV^2}{2g}, \text{ i.e., } \frac{4u(V-u)}{V^2}.$$

This is unity when $u = \frac{1}{2}V$.

The water exerts a mean tangential pressure P on the buckets which can be thus found: let s be the area of the cross-section of the jet; then, measuring backwards from the wheel along the jet a length of column equal to $(V-u)\,\tau$, the volume $(V-u)\,\tau s$ of water will strike the wheel in time τ; the absolute velocity of this water is changed from V to $2u-V$, $\therefore$ its change of momentum is, if w is the weight per unit volume, $\quad 2ws\tau(V-u)^2$,

and by the equation of impulse and momentum,

$$\frac{2ws\tau(V-u)^2}{g} = P\tau.$$

$$\therefore \; P = \frac{2ws(V-u)^2}{g}.$$

The Centrifugal Pump. Imagine the hollow central axis of the turbine wheel shown in Fig. 83 to be connected with a vertical pipe dipping into a well at a depth of about 20 feet, or less, below the centre of the wheel, and that the pipe P is connected with a vertical pipe bent at its upper end over a tank. Suppose also that the whole of the pump chambers and the pipe dipping into the well are completely full of water. If the atmosphere is excluded from the pump chambers, the pressure at the centre of the wheel is less than atmospheric by the pressure due to the height of the centre of the wheel above the well.

Now suppose the wheel to be driven by some engine coupled with the driving pulley of the wheel in the sense opposite to that represented in the figure. Then the water in the chambers is set in motion, and consequently pressure diminishes within it, and water ascends from the well. Thus the whirlpool chamber and the pipe leading to the elevated tank get continuous supplies of water raised from the well, and the machine acts as a force pump for filling the tank. A turbine driven backward in this way is called a *centrifugal pump*. Such pumps are frequently used in pumping water out of the boiler compartments of large war ships, and also for drainage and for emptying docks.

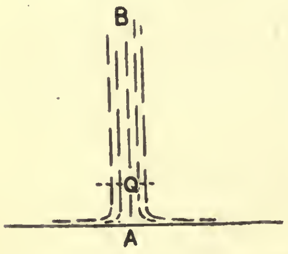


Fig. 87.

49. Pressure of a jet on a plane. Let AB, Fig. 87, represent a column of fluid striking a fixed plane at A, and let v be the velocity of the particles in the neighbourhood of A. Then if the stream flows continuously, the effect on the plane is a continuous pressure. The magnitude of this we propose to find.

Measure backwards along the stream a length AQ equal to $v.\Delta t$, where Δt is any infinitesimally small time; then all the particles in the column between A and Q will strike the plane in the time Δt. If S is the area of the cross-section of the stream and w the weight per unit volume of the fluid, the weight of the column AQ is $wSv\,.\,\Delta t$, and its momentum is $wSv^2\,.\,\Delta t$. This momentum is destroyed by the plane in the time Δt; and if P is the force, in gravitation units, capable of destroying the momentum,

$$P\Delta t = \frac{wSv^2}{g}\Delta t,$$

$$\therefore \quad \frac{P}{S} = \frac{wv^2}{g};$$

and $\frac{P}{S}$ is the force, or pressure, per unit area on the plane.

If the velocity v is such as might be acquired in falling through a height h, and $\frac{P}{S}$ is denoted by p, we have

$$p = 2wh,$$

which is the intensity of pressure that would be produced by a statical column of the liquid of height $2h$. If the stream is produced by a waterfall of height h, then a statical column of double this height would produce the same effect on the plane.

This result holds, with some modification, for the intensity of pressure produced on the wall of a vessel containing a gas. At each point inside the vessel the gas particles are moving in all possible directions with very various velocities. Now these velocities give a certain *mean square* of their values, which mean square remains constant at all points of the gas so long as the temperature remains constant. Let $\bar{v}^2$ be the value of the mean of the squares of the velocities of the gas particles at a point P, velocities being taken in all directions round P. Now supposing that we fix on a single fixed direction, Px, at P, and consider only components of velocities measured in that direction, what will be the mean square of these at any instant? Evidently much less than $\bar{v}^2$, since in finding this the whole of each velocity is taken and not merely a component of it. It is proved in works on the kinetic theory of gases that the mean square of components in a single direction Px is just one-third of the total mean square, so that if we denote the former by $\bar{v}_x{}^2$, $\bar{v}_x{}^2 = \frac{1}{3}\bar{v}^2$.

The reasoning which applied to the unidirectional stream BA, Fig. 87, can now be applied to the gas, and we have

$$p = \tfrac{1}{3} \cdot \frac{w\bar{v}^2}{g}$$

for the intensity of pressure exerted on the wall of the vessel.

In this expression w is the weight of a unit volume of the gas at the temperature of the gas. Suppose that we use the C. G. S. system, and employ the formula for the weight given in Chap. V. If k is a numerical constant, we know that $w = k\frac{p \cdot s}{T}$, where s and T are the specific gravity and absolute temperature of the gas. Hence the above equation gives

$$\bar{v} = \mu\sqrt{\frac{T}{s}},$$

where μ is a numerical constant, and $\bar{v}$ is the square root of the mean square of the velocities of particles. This $\bar{v}$ is called the 'velocity of mean square'. With the numbers for the metric system, if velocity is measured in metres per second and s is the specific gravity of the gas referred to hydrogen (about 14·4 times that referred to air), we find

$$\bar{v} = 111{\cdot}4\sqrt{\frac{T}{s}}.$$

EXAMPLES.

1. Water enters a horizontal pipe at an end where the area of the cross-section is A square feet; the pipe contracts to a cross-section B, and after this changes to a cross-section C at the other end, where the water flows into the atmosphere, K cubic feet being delivered per second; find the pressure intensities at A and B.

Result. If w = weight of a cubic foot of water, the pressure at A is $p_0 + \frac{wK^2}{2g}\left(\frac{1}{C^2} - \frac{1}{A^2}\right)$; that at B is $p_0 + \frac{wK^2}{2g}\left(\frac{1}{C^2} - \frac{1}{B^2}\right)$, where p_0 (per square foot) is the atmospheric pressure.

2. A turbine whose blades are radial at the outer rim receives 40,000 lbs. of water per minute; the outer radius of the wheel is 1 foot, and the turbine is to have a horse-power of 50; how many revolutions per minute must the wheel make?

Result. If v feet per sec. is the circumferential velocity of the wheel, we have $\frac{40000}{60} \cdot \frac{v^2}{32} \cdot \frac{1}{550} = 50$, $\therefore\ v = 36{\cdot}3$ *f/s*, and the number of revolutions per minute $= \frac{36{\cdot}3}{2\pi} \times 60 = 347$.

3. A turbine whose blades are radial at the outer rim receives W lbs. of water per minute; the outer radius of the wheel is a feet, and the machine is to develop m horse-power; how many revolutions must the wheel make?

Result. $\frac{9815}{a}\sqrt{\frac{m}{W}}$.

4. The inner and outer radii of a turbine wheel are b and a; the velocity of flow through the inner circumference is $\frac{1}{8}$ of that due to the head of water, and the outer circumferential velocity is due to half the head; find the inclination of the vanes to the inner circumference.

Result. $\tan^{-1}\frac{a\sqrt{2}}{8b}$.

5. The blades of a turbine at its outer circumference are radial; 500 lbs. of water per second are supplied, with a head

of 28 feet; find the total area of the inlet surface of the wheel if the radial velocity at inlet is $\frac{1}{8}$ of that due to the head, taking $g = 32$.

Result. $\frac{4}{7}\sqrt{7}$ square feet.

6. If the velocity of the above turbine wheel at its outer circumference is 40 *f/s*, the head of water 49 feet, the radial velocity at inlet $\frac{1}{8}$ of that due to the head, find the magnitude and direction of the absolute velocity of the water at inlet.

Result. $\sqrt{40^2+6^2}$, $\tan^{-1}(0{\cdot}15)$.

7. The outer radius of the wheel of a centrifugal pump is a feet; a turning-moment of L foot-pounds' weight is applied to the wheel spindle, and the pump raises W lb. per second; find the tangential velocity of discharge of the water.

Result. $\frac{gL}{Wa}$.

50. Fluid revolving about vertical axis. If a vessel, represented in a vertical section by ACB, Fig. 88, and containing a fluid, is set rotating round a vertical axis, Cz, after a short time the fluid, owing to friction between its particles and against the surface of the vessel, will rotate like a rigid body with the angular velocity ω; each particle, P, will describe a horizontal circle with this angular velocity, so that if PN is the perpendicular from P on the axis of rotation, the resultant acceleration of the particle is directed along PN from P towards N, and is $\omega^2 . NP$ in magnitude. Denote NP by r, and consider an indefinitely small particle of mass dm at P; then the resultant mass-acceleration of this particle is

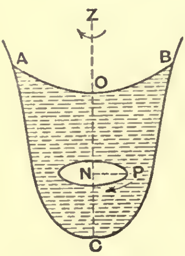


Fig. 88.

$$\omega^2 r \, . \, dm, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and this vector is directed from P towards N. [The *reversed* mass-acceleration, $-\omega^2 rdm$, is called the *force of inertia* of the particle, or its *resistance to acceleration*. It is most important to understand that this force of inertia is not a force acting *on* the particle, but one exerted *by* it on the surrounding medium, or, generally, on the agent or agents accelerating its motion. Thus, then, if a is the vector representing the resultant acceleration of a particle, dm, a force completely represented by

$$-a \,.\, dm \text{ in absolute units, or}$$

$$-\frac{a}{g} \,.\, dm \text{ in gravitation units,}$$

is the resultant force exerted *by* the particle *on* the agents acting upon it.

Now the fundamental principle of all Dynamics is this: for *each* particle of any material system (whether rigid body, natural solid, liquid, or gas) the resultant mass-acceleration is in magnitude and direction the exact resultant of *all* the forces acting upon the particle. These forces will, in general, consist partly of pressures from the surrounding particles, partly of attractions from these particles, and partly of attractions from bodies outside the system. If we consider the equivalence of the resultant mass-acceleration, $a\,dm$, to these forces under three separate heads, we deduce three great principles of Dynamics. Thus, if we consider that $a\,dm$ has—

1. The same virtual work for any imagined displacement of the particle,

2. The same moment about any axis,

3. The same component along any line,

as the whole system of forces acting on the particle, and that this is true for *every* particle of the system, we have at once the principles of—

1. Kinetic Energy and Work,

2. Time-rate of change of Moment of Momentum,

3. Motion of Centre of Mass,

for every material system. If the forces acting are measured in gravitation units, their complete equivalent is $\frac{a}{g}\, dm$.]

Suppose that at any point, P, Fig. 89, we take as the element dm a very short and thin cylinder, *abcd*, of the fluid having its axis along the tangent at P to any curve AB. Let the length $bc = ds$; let σ be the area of the cross-section, *ab*, of the cylinder; let F be the external force per unit mass exerted on the fluid at P, and therefore Fdm the force on the cylinder, not including the pressure exerted on its surface by the surrounding fluid; let a be the resultant acceleration of the particle; and let p be the intensity of pressure on the face *ab*, and therefore $p + \frac{dp}{ds} ds$ that on *cd*.

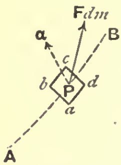


Fig. 89.

Then $dm = w\sigma ds$, if w is the mass of the fluid per unit volume at P; and if we resolve forces in the direction of the tangent at P to the curve, we see that $w\sigma ds \cdot \frac{a}{g}$ has the same component along this tangent, in the sense PB, as $F \cdot w\sigma ds$ and $-\frac{dp}{ds} ds \cdot \sigma$, the length of the arc s being measured from A. Hence if a_s is the component of a along the tangent at P and S is the component of F, we have

$$\frac{a_s}{g} = S - \frac{1}{w}\frac{dp}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation connects the acceleration in any direction with the force intensity and the rate of change of pressure intensity in that direction.

Now suppose the external force to be gravity. Taking the right line NP (Fig. 88) as the direction of s, (2) becomes

$$w \cdot \frac{\omega^2 r}{g} = \frac{dp}{dr}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and again, taking the vertical downward direction at P as that of s, (2) becomes

$$0 = w - \frac{dp}{dz}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where z is the depth of P below any fixed horizontal plane.

Now p is a function of r and z only, so that

$$dp = \frac{dp}{dr}dr + \frac{dp}{dz}dz$$

$$= w\left(\frac{\omega^2 r}{g}dr + dz\right) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\therefore \quad p = w\left(\frac{\omega^2 r^2}{2g} + z\right) + C,$$

where C is a constant, which may be determined from a knowledge of p at some one point. If p_0 is the value of p at O, the point in which the free surface cuts the axis of rotation, and if O is taken as origin, since $r = z = 0$ at O, we have $C = p_0$; hence

$$p = p_0 + w\left(\frac{\omega^2 r^2}{2g} + z\right). \quad . \quad . \quad . \quad . \quad . \quad (6)$$

At all points on the free surface $p = p_0$, therefore the equation of this surface is

$$\frac{\omega^2 r^2}{2g} + z = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

showing that the z of every point on it is negative, i.e., all these points are higher than O. This equation denotes a parabola whose latus rectum is $\frac{2g}{\omega^2}$, and the free surface is therefore a paraboloid generated by the revolution of this surface round Oz.

If the vertical upward line Oz is taken as axis of x, and a tangent at O as axis of y, the equation of the parabola is, in its usual form,

$$y^2 = \frac{2g}{\omega^2} \cdot x. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the fluid contained in the vessel is a gas, equations (2), (3), (4) still hold, and, in addition, $p = kw$ (Art. 33); hence (5) becomes

$$\frac{dp}{p} = \frac{1}{k}\left(\frac{\omega^2}{g} rdr + dz\right), \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\therefore \quad p = Ae^{\frac{1}{k}\left(\frac{\omega^2 r^2}{2g} + z\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where A is a constant to be determined either from a knowledge of p at some point or from the given mass of the fluid. Equation (10) shows that for a gas the free surface and the surfaces of constant pressure intensity are still paraboloids.

In the same way, if the vessel contains two fluids that do not mix, their surface of separation is a paraboloid of revolution. For if w, w' are their specific weights, we have (if they are liquids)

$$p = w\left(\frac{\omega^2 r^2}{2g} + z\right) + C \text{ for one,}$$

$$p' = w'\left(\frac{\omega^2 r^2}{2g} + z\right) + C' \text{ for the other,}$$

and since at all points on the surface of separation $p = p'$, we have the equation of a paraboloid of revolution, as before.

Examples.

1. A cylinder contains a given quantity of water. If it is rotated round its axis (held vertical), find the angular velocity at which the water begins to overflow.

Let AOB represent the surface of the rotating liquid, the points A and B being at the top of the cylinder; let r and h be the radius and height of the cylinder, and c the height to which the cylinder, when at rest, is filled.

Then since B is on the parabola, if ξ is the depth of O below AB,

$$r^2 = \frac{2g}{\omega^2} \cdot \xi. \quad \dots\dots \quad (1)$$

But the volume of the water remains unchanged, and it is the volume of the cylinder minus the volume of the paraboloid AOB. This latter is $\frac{\pi g}{\omega^2} \cdot \xi^2$. Hence

$$r^2 h - \frac{g}{\omega^2}\xi^2 = r^2 c, \quad \dots\dots \quad (2)$$

$$\therefore \quad \omega = \frac{2\sqrt{g(h-c)}}{r}. \quad \dots\dots \quad (3)$$

The above is on the supposition that the water begins to overflow before the vertex, O, of the parabola reaches the base, C, of the cylinder. In this case, with any angular velocity, ω, if PQ is the level to which the water rises, and LM is the level at which the water stands when at rest, it is easily proved that

depth of O below LM = height of PQ above LM. . (4)

Take now the case in which c is so small in comparison with h that O reaches C (or the base begins to get dry) before the water begins to overflow. The angular velocity at which O reaches C is $\frac{2\sqrt{gc}}{r}$. Let $\omega = (1+n)\frac{2\sqrt{gc}}{r}$; then if O' (below C) is the vertex of the parabola, we have

$$CO' = 2n(1+n)c, \quad \dots\dots \quad (5)$$

the height of PQ (the water level) above the base is $2c(1+n)$; and if the free surface cuts the base in R, we have

$$RC = r\sqrt{\frac{n}{1+n}}, \quad \dots\dots \quad (6)$$

which is the radius of the dry circle on the base. The water will begin to overflow when the height of PQ above the base is h; i.e.,

$$\omega = \frac{h}{r}\sqrt{\frac{g}{c}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which is quite different in form from (3); so that if c is infinitely small, i.e., if there is only an infinitely thin layer of water put originally into the cylinder, it will not begin to overflow until ω is infinitely great; and in this case $CR = r$, as it should be.

2. A heavy cylinder floats with its axis vertical in a liquid contained in a vessel which rotates uniformly round a vertical axis; find the length of the portion of the cylinder immersed.

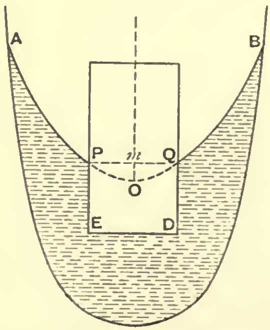


Fig. 90.

Let PQ, Fig. 90, be the level of the liquid round the cylinder, and $PEDQ$ the immersed portion, the free surface being $APOQB$, and O the vertex of the parabola.

Now, by the same reasoning as that in p. 91, it is obvious that the resultant action of the liquid on the cylinder is the same as that which the liquid would exert on the liquid which would fill the volume $POQDEP$; hence the weight, W, of the cylinder must be equal to the weight of this volume of the liquid.

Let r be the radius of the cylinder and Om the perpendicular from O on PQ; then $Om = \frac{\omega^2 r^2}{2g}$, and the volume of the displaced liquid

$$= \pi r^2 . PE - \text{vol. } POQ = \pi r^2 \left(PE - \frac{\omega^2 r^2}{4g}\right);$$

hence if w = specific weight of liquid,

$$PE = \frac{W}{\pi r^2 w} + \frac{\omega^2 r^2}{4g}.$$

3. A vessel of given form containing water is set rotating round a vertical axis, the vessel and the liquid being in relative equilibrium; find the greatest angular velocity of the vessel which will allow all the water to escape through a small orifice at the lowest point of the vessel.

Let the vessel be ACB, Fig. 88, C being its lowest point; assume the free surface to pass through C, the latus rectum of the parabola being $\frac{2g}{\omega^2}$; then, taking C as origin, the tangent at C as axis of y and the vertical upward line as axis of x, express the condition that the parabola $y^2 = \frac{2g}{\omega^2}x$ intersects the curve ACB in no other point than C.

Thus, if the vessel is a sphere (with another small hole at the top) of radius a, the parabola will be completely outside the sphere if $\omega^2 = \frac{g}{a}$.

4. A cylinder whose axis is vertical is filled with a given mass of gas and set rotating round its axis; if the gas is assumed to move in relative equilibrium with the cylinder, find the intensity of pressure at any point.

We have, measuring z from the top of the cylinder

$$dp = wd\left(\frac{\omega^2 r^2}{2g} + z\right),$$

$$\therefore\ p = Ae^{\frac{1}{k}\left(\frac{\omega^2 r^2}{2g} + z\right)},$$

$$\therefore\ w = \frac{A}{k}e^{\frac{1}{k}\left(\frac{\omega^2 r^2}{2g} + z\right)}.$$

Also if W is the weight of the gas put into the cylinder, we have

$$W = \int wd\,\Omega,$$

where $d\Omega$ = element of volume at any point, P (Fig. 88). Now if θ is the angle which the plane of P and the axis, Oz, of rotation makes with any fixed vertical plane,

$$d\Omega = rd\,\theta\,dr\,dz;$$

$$\therefore\ W = \frac{2\pi A}{k}\iint e^{\frac{1}{k}\left(\frac{\omega^2 r^2}{2g} + z\right)} r\,dr\,dz.$$

Integrating with respect to r, the limits of r are 0 and a, where a is the radius of the cylinder, so that the integrations in r and z may be performed independently, the limits of z being 0 and h. We easily find

$$W = \frac{2\pi g k A}{\omega^2}\left(e^{\frac{\omega^2 a^2}{2gk}} - 1\right)\left(e^{\frac{h}{k}} - 1\right),$$

which determines A.

5. If the cylinder is replaced by a spherical shell rotating about a vertical diameter, solve the previous problem.

6. A hemispherical bowl containing a given quantity of water is set rotating about a vertical diameter, find the angular velocity at which the water begins to overflow.

Result. If V is the volume of the water, a the radius of the bowl,

$$\omega^2 = \frac{4g}{\pi a^4}\left(\tfrac{2}{3}\pi a^3 - V\right).$$

7. If in the last case the angular velocity is increased beyond the value $\left(\frac{2g}{a}\right)^{\frac{1}{2}}$, find how much of the bowl is dry.

Result. It is dry to a vertical height $a - \frac{2g}{\omega^2}$ above the lowest point.

8. If a hollow open cone with its axis vertical and vertex downwards containing a given quantity of water is made to revolve round a vertical axis, discuss the question as to the possibility of emptying the cone by increasing the angular velocity.

9. A narrow horizontal tube, BC, has two open vertical branches BA and CD, water being poured into the continuous tube, thus formed, to a given height. If this tube is set rotating round a vertical axis through a point O in BC, find the position of the liquid in its state of relative equilibrium.

Result. If $BO = m$, $OC = n$, the difference of level in the two vertical branches is $\frac{\omega^2}{2g}(m^2 - n^2)$.

10. A straight tube, AB, containing water is held inclined at an angle θ to the vertical line at A, and the tube is set in rotation with given angular velocity, ω, about this vertical line; find the greatest length of column of liquid so that there shall be no separation of the column anywhere.

Let x be the length of column; let P be a point in it such that r and z are the distances of P from the axis of rotation and the horizontal line through B; then

$$p = w\left(\frac{\omega^2 r^2}{2g} + z\right) + C.$$

At A we have $p = wh$, where h is the height of a water barometer, and if l is used for $\frac{g}{\omega^2}$,

$$\frac{2l}{w} \cdot p = r^2 - 2lr\cot\theta + 2l(h + x\cos\theta) - x^2\sin^2\theta.$$

Now if P is zero anywhere in the column, the column breaks there. Make it impossible for p to vanish for any value of r, and we have

$$2l(h + x\cos\theta) - x^2\sin^2\theta > l^2\cot^2\theta.$$

If we make these two sides equal, we have the limiting value of x, viz.

$$\frac{l\cos\theta + \sqrt{2hl}\sin\theta}{\sin^2\theta}.$$

11. If in Q. 9 the lengths of BA and CD are each c and the liquid reaches to their tops, and if O is the middle point of BC, prove that no liquid will overflow until the angular velocity exceeds

$$\frac{\sqrt{2g(h+c)}}{a},$$

where h is the height of a barometer formed of the liquid, and $BC = 2a$.

Taking O as origin, z being measured upwards from OC,

$$p = w\left(\frac{\omega^2 r^2}{2g} - z\right) + C,$$

and since at D we have $p = wh$, we have

$$p = w\left[h - \frac{\omega^2}{2g}(a^2 - r^2) - z + c\right].$$

Hence at O

$$p = w\left(h - \frac{\omega^2}{2g}a^2 + c\right),$$

and when the liquid begins to overflow, the horizontal column splits at O, $\therefore p = 0$ at O. Hence the result.

12. In the last if BC alone is filled with liquid, find the angular velocity which will cause the liquid to rise to a height k in the vertical branches.

Result. $\omega^2 = 2g\,\dfrac{h+k}{a^2 - k^2}$.

13. If in the last the upper ends of BA and CD are closed before the motion begins, these branches containing air at atmospheric pressure, find the speed necessary to raise the liquid to a height k in the vertical branches.

Result. $\omega^2 = 2g\dfrac{k + \dfrac{ch}{c-k}}{a^2 - k^2}$.

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