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A TREATISE ON
HYDROSTATICS

VOL. II

CONTAINING THE MORE ADVANCED
PART OF THE SUBJECT

BY

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PREFACE TO VOL. II

THIS volume contains that portion of the subject which is suitable to students who can apply a knowledge of advanced mathematics.

The first edition of this work contained certain chapters which are omitted in this edition—e.g. those on thermodynamics and wave motion under the action of gravity—inasmuch as students are likely to study these subjects at greater length in special treatises.

The treatment of the general equations of pressure, of bodies floating freely and under constraint, and of the ellipsoidal figures of a revolving self-attracting liquid, has, however, been considerably enlarged. New graphic constructions for the Maclaurin and Jacobi ellipsoids (exhibited at a meeting of the Oxford Mathematical and Physical Society) are also introduced.

I have again to acknowledge the assistance kindly given to me by Mr. Pidduck in the work of revision.

GEORGE M. MINCHIN.

OXFORD,
September, 1912.

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CHAPTER I

CENTRES OF PRESSURE

51. CONFINING our attention still to the case in which pressure in a fluid is due to gravity only, when the contour of a plane area consists of a curve, and not merely of straight lines, something more than the simple rules of the preceding section is required. We have usually in this case to break up the area into narrow horizontal strips, apply the principle of mass-moments, and integrate.

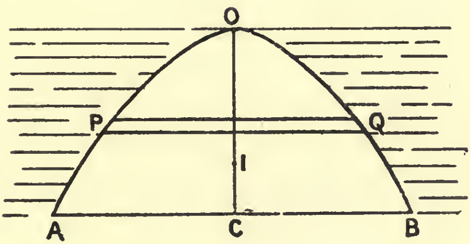

Fig. 1.

For example, suppose a plane area bounded by a parabola and a double ordinate to be immersed vertically in a liquid with the vertex in the surface of the liquid and the ordinate horizontal; it is required to find the position of the centre of pressure. Divide the area into narrow horizontal strips, such as PQ (Fig. 1). Let x be the depth of PQ, dx its breadth and $y^2 = 4ax$ the equation of the parabola.

Then the pressure on PQ is

$$2wydx \times x;$$

and its force-moment about the horizontal plane at O is

$$2wx^2ydx.$$

Hence the total moment of pressure about the plane is

$$2w\int_0^h x^2ydx,$$

if h is the height of the area, or the depth of AB below O.

This moment $= 4wa^{\frac{1}{2}}\int_0^h x^{\frac{5}{2}}dx = \frac{8}{7}wa^{\frac{1}{2}}h^{\frac{7}{2}}$; and the whole pressure on the area

$$= 2w\int_0^h xydx = 4wa^{\frac{1}{2}}\int_0^h x^{\frac{3}{2}}dx = \frac{8}{5}wa^{\frac{1}{2}}h^{\frac{5}{2}};$$

and if z is the depth of its point of application,

$$\frac{8}{5}wa^{\frac{1}{2}}h^{\frac{5}{2}}.z = \frac{8}{7}wa^{\frac{1}{2}}h^{\frac{7}{2}},$$

$$\therefore \quad z = \frac{5}{7}h.$$

Since the pressure on each strip acts at its middle point, and all the middle points lie on the vertical OC, the centre of pressure, I, is on OC. If the base, AB, of the area is in the surface and O below, x being still the distance of PQ from O, the pressure on the strip is

$$2wy(h-x)dx,$$

and its moment about the surface is

$$2wy(h-x)^2dx;$$

so that if z is the depth of the centre of pressure,

$$z\int_0^h y(h-x)dx = \int_0^h y(h-x)^2dx,$$

$$\therefore \quad z = \frac{4}{7}h.$$

In this case we may, of course, take force-moments about the horizontal plane through O instead of about the surface; and if z' is the distance of the centre of pressure from O,

$$z'\int_0^h y(h-x)\,dx = \int_0^h y(h-x)\,x dx,$$

$$\therefore \quad z' = \tfrac{3}{7}h.$$

Force-moments may be taken about any plane whatever, but the depths on which the values of the pressures depend must, of course, be measured from the free surface of the liquid.

As another example take a board AOD (Fig. 2) in the form of a quadrant of a circle immersed vertically in a liquid with one of its bounding radii, OA, in the surface. It is required to find the position of the centre of pressure.

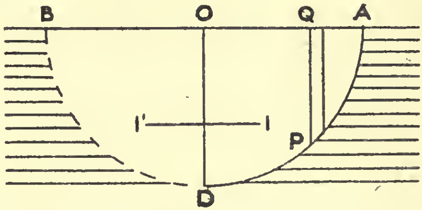

Fig. 2.

The area may be divided either into vertical or into horizontal strips. Suppose the former. Let PQ be an ordinate, y, and OQ the corresponding abscissa, x. The pressure on the strip is

$$wydx \cdot \frac{y}{2},$$

and it acts at a depth $\frac{2}{3}y$; hence its moment about the surface OA is

$$\tfrac{1}{3}wy^3dx;$$

and if z is the depth of the centre of pressure below OA,

$$z \cdot \tfrac{1}{2}w\int_0^r y^2dx = \tfrac{1}{3}w\int_0^r y^3dx,$$

the value of x running from 0 to r, where $r = OA$.

If θ is the angle POA, $x = r\cos\theta$ and $y = r\sin\theta$, so that

$$z\int_0^{\frac{\pi}{2}} \sin^3\theta d\theta = \tfrac{2}{3}r\int_0^{\frac{\pi}{2}} \sin^4\theta d\theta,$$

$$\therefore \quad z = \frac{3\pi}{16}r.$$

This does not determine the position of the centre of pressure; it gives merely its distance from OA. Its distance from OD must also be found. Taking force-moments about the vertical plane through OD, if z' is the distance of I from OD,

$$z' \cdot \tfrac{1}{2}w\int y^2dx = \tfrac{1}{2}w\int xy^2dx,$$

since the distance of the point of application of the pressure on the strip PQ from OD is x. Again using the angle θ, we have

$$z'\int \sin^3\theta d\theta = r\int \cos\theta \sin^3\theta d\theta,$$

$$\therefore \quad z' = \tfrac{3}{8}r.$$

It is to be observed that the depth of the centre of pressure on the quadrant AD is the same as that of the centre of pressure on the semicircular board ADB, since the centre of pressure on the latter lies on OD and also on the line II' joining the centres of pressure on the two quadrants.

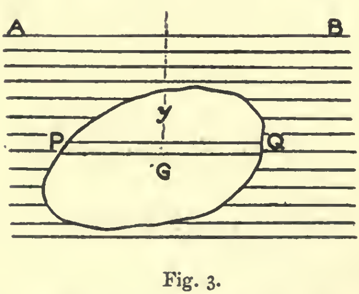

Fig. 3.

If a plane area of any form is immersed vertically in a liquid, the depth of the centre of pressure below the surface of the liquid is $\frac{k^2}{\bar{z}}$, where k is the radius of gyration of the area about the line AB (Fig. 3) in which the plane of the area cuts the surface of the liquid, and $\bar{z}$ is the depth of the centre of gravity G, of the area.

Divide the area into horizontal strips by lines such as PQ; let y be the depth of PQ, and let $PQ = \beta$. Then the pressure on the strip is $w \cdot \beta dy \cdot y$, and its moment about AB is $w \cdot \beta y^2 dy$. The total moment of pressure is $w \int \beta y^2 dy$; but $\int \beta y^2 dy = Ak^2$, where A is the whole area. Also the total pressure is (Art. 12) $A\bar{z}w$. Hence if ζ is the depth of the centre of pressure,

$$A\bar{z}w \cdot \zeta = w \cdot Ak^2,$$

$$\therefore \quad \zeta = \frac{k^2}{\bar{z}}, \quad \ldots \ldots \quad (a)$$

which result is often very convenient. Thus, in the case of the semicircle ADB, Fig. 2, since the distance of G from O is $\frac{4r}{3\pi}$, and k^2 about AB is $\frac{r^2}{4}$, we have the depth of I as before.

The result (a) does not, of course, determine the position of the point completely.

It is well to point out that the position of the centre of pressure on a plane area is unaltered if the area, instead of being vertical, is rotated about the line AB in which its plane cuts the surface of the liquid; for, if y still denotes the distance of a strip PQ from AB, the depth of the strip below the surface of the liquid is $y \sin\theta$, where θ is the angle made with the surface of the liquid by the plane. Hence the pressure on the strip is $w \cdot \beta dy \cdot y \sin\theta$, and the moment of this about AB (that is, about a plane through AB perpendicular to the plane of the rotated area) is

$$w \cdot \beta dy \cdot y \sin\theta \cdot y, \quad \text{or} \quad w\beta y^2 \sin\theta \, dy.$$

Hence if ζ is the distance of the centre of pressure from AB,

$$\zeta . \int w . \beta dy . y \sin\theta = \int w \beta y^2 dy \sin\theta$$

$$\text{or } \zeta = \frac{\int \beta y^2 dy}{\int \beta y dy} = \frac{k^2}{\bar{z}},$$

since θ is a constant for all the strips.

In the same way if any line LM is drawn perpendicular to AB in the plane of the revolving area, and if ξ is the distance of the mid point of the strip PQ from LM, the moment about LM of the pressure on the strip is

$$w \beta y^2 \sin\theta \, dy \times \xi ;$$

and the whole moment of pressure is

$$w \sin\theta \int \beta y^2 \xi dy.$$

Dividing this by the whole pressure,

$$w \sin\theta \int \beta y^2 dy,$$

we have the distance of the centre of pressure from LM; and this again is independent of θ. Whatever be the angle, therefore, through which the area turns round AB, the centre of pressure remains fixed in the area.

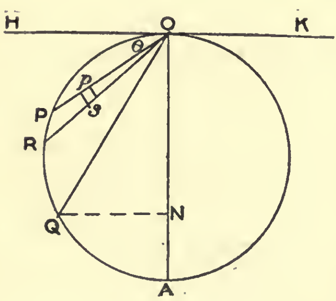

Fig. 4.

It is sometimes desirable to use polar co-ordinates in calculations relating to pressure. Thus, suppose that a circular area is immersed vertically in a liquid with its highest point, O, Fig. 4, in the surface, HK, of the liquid, and that the position of the centre of pressure on a segment OPQ is required. Take any point p in the area; let $pO = r$, $\angle pOH = \theta$, and describe the small element ps of area, which is $rdrd\theta$, formed in the usual way by taking two close values of r and of θ. Then the pressure on ps is $wrdrd\theta \,.\, r\sin\theta$; and the moments of this about OH and OA (the vertical diameter) are

$$wr^3 \sin^2\theta\, drd\theta \quad \text{and} \quad wr^3 \sin\theta \cos\theta\, drd\theta.$$

We shall first integrate over the triangular strip POR; that is, we keep θ constant and let r run from 0 to OP, or from 0 to $2a\sin\theta$, where a is the radius of the circle. Thus we get for the moment about OH

$$w \sin^2\theta\, d\theta \int_0^{2a\sin\theta} r^3 dr, \text{ or } 4wa^4 \sin^6\theta \,.\, d\theta.$$

Hence the whole moment is

$$4wa^4 \int_0^{\alpha} \sin^6\theta\, d\theta,$$

where $\alpha = \angle\, QOH$. The total pressure on the area OPQ is

$$\tfrac{8}{3} wa^3 \int_0^{\alpha} \sin^4\theta\, d\theta;$$

and if ζ is the distance of the centre of pressure from OH,

$$\zeta = \tfrac{3}{2} a \frac{\int_0^{\alpha} \sin^6\theta\, d\theta}{\int_0^{\alpha} \sin^4\theta\, d\theta}.$$

Similarly, integrating the moment about OA of the pressure on ps, if ξ is the distance of the centre of pressure from OA,

$$\xi = \tfrac{1}{4} a \frac{\sin^6\alpha}{\int_0^{\alpha} \sin^4\theta\, d\theta}.$$

The integrals are easily found by the usual methods of the Calculus.

The result can also be obtained by drawing the ordinate

QN and considering the pressure on the area $OPQN$ as the resultant of that on the segment OPQ and that on the triangle OQN.

Examples.

1. A circular area is immersed in water with its highest point in the surface; show that a horizontal line drawn across the area at a depth equal to $\frac{3}{4}$ of the diameter divides the area into two parts on which the pressures are in the ratio 2 : 1.

2. Find the position of the centre of pressure on the upper of the above parts of the circle.

Result. $\left(\frac{5}{4} - \frac{27\sqrt{3}}{64\pi}\right)r.$

3. An area bounded by the curve $cy^2 = x^3$ and a double ordinate at a distance h from the origin is placed vertically in water with the double ordinate in the surface; find the position of the centre of pressure.

Result. On the axis of x at a distance $\frac{5}{9}h$ from the origin.

4. If in the last case O is the origin, PQ the double ordinate, and A the point in which PQ is intersected by the vertical through O (axis of x), find the position of the centre of pressure on the area OAQ of the curve.

Result. Its co-ordinates with respect to O are

$$\tfrac{7}{32}AP,\ \tfrac{5}{9}OA.$$

5. If in the last case a parabola is described having its vertex at O, its axis along OA, and passing through the point P, find the position of the centre of pressure on the loop OP included between the two curves.

Result. If ξ, η are the co-ordinates of the centre of pressure referred to the axes at O, $\xi = \frac{1}{3}OA$, $\eta = \frac{49}{128}AP$.

(Consider the pressure on the parabolic area as the resultant of that on the loop and that on the semi-cubical parabola $cy^2 = x^3$.)

6. A plane area bounded by a cycloid and its base AB is immersed vertically in a liquid with the base in the surface

and the vertex, O, beneath; C is the middle point of AB. Find the position of the centre of pressure on that part of the area which lies at one side between the cycloid and the circle having CO for diameter.

Result. If $CO = 2a$, the depth of the centre of pressure is $\frac{7}{6}a$, and its distance from CO is $\frac{45\pi^2 - 128}{90\pi}a$.

7. A plane area bounded by the lemniscate $r^2 = a^2 \cos 2\theta$ is placed with the origin of r in the surface of a liquid and its axis vertical; find the position of the centre of pressure on one half of the area made by the axis.

Result. At a depth $\left(\frac{1}{4} + \frac{2}{3\pi}\right)a\sqrt{2}$, and distant $\frac{\sqrt{2}}{3\pi}a$ from the axis.

52. Centre of Pressure referred to principal axes of area. The position of the centre of pressure on a plane area can be very easily expressed with reference to the principal axes of the area at its centre of gravity, G. Thus, let CDE (Fig. 5) be the plane area, its plane being inclined to the vertical at any angle, θ; let GA and GB be its principal axes at G, intersecting the surface of the liquid in A and B; let GN $(= h)$ be the perpendicular from G (in the plane of the area) on the line AB, and let GN make the angle α with GA.

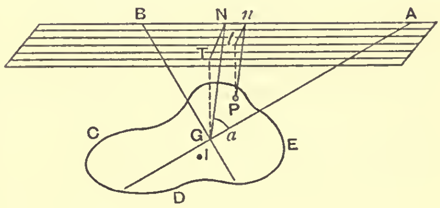

Fig. 5.

The equation of AB with reference to GA and GB as axes of x and y, respectively, is

$$x \cos \alpha + y \sin \alpha - h = 0;$$

and if P is any point in the area at which the element of area dS is taken, the perpendicular Pn from P on AB is $h-x\cos a-y\sin a$, if x, y are the co-ordinates of P with reference to GA and GB. Hence the perpendicular, Pt, from P on the surface of the liquid is

$$(h-x\cos a-y\sin a)\cos\theta,$$

and the pressure on dS is

$$w(h-x\cos a-y\sin a)\cos\theta\,.\,dS, \quad . \quad . \quad . \quad (a)$$

where w is the weight of the liquid per unit volume. Now take the sum of the moments of the elementary pressures of which (a) is the type about GA and equate it to the moment of the resultant pressure, $A\times GT\,.\,w$, where A is the area and GT the perpendicular from G on the surface. If (ξ, η) are the co-ordinates of I, the centre of pressure, we have

$$wAh\cos\theta\,.\,\eta = w\cos\theta\int(h-x\cos a-y\sin a)\,ydS \quad . \quad (2)$$
$$= -w\cos\theta\sin a\int y^2dS,$$

the other integrals vanishing since the principal axes at the centre of area are those of co-ordinates. Now $\int y^2dS$ is the moment of inertia of the area about GA, which we shall denote by $A\,.\,k_1^2$, k_1 being the radius of gyration of the area about GA. Hence, finally,

$$\eta = -\frac{k_1^2}{h}\sin a\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and in the same way, equating the moment of the whole pressure about GB to the sum of the moments of the elementary pressures of the type (a), we have

$$\xi = -\frac{k_2^2}{h}\cos a. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus the co-ordinates are independent of the inclination of the given plane area to the vertical, as we have previously pointed out, so that if the area were turned round the line

AB in which its plane intersects the surface of the liquid through any angle, the centre of pressure, I, would continue to be absolutely the same point in the area.

The expressions (3), (4) lead at once to an obvious geometrical interpretation, viz.—construct the ellipse whose equation with reference to GA and GB is

$$\frac{x^2}{k_1^2} + \frac{y^2}{k_2^2} = 1 ;$$

take the pole, Q, of the line AB with reference to this ellipse; the co-ordinates of Q are $(-\xi, -\eta)$, so that if the line QG is produced through G to I so that $GI = QG$, we arrive at I, the centre of pressure.

These expressions (3), (4) give us at once some simple results concerning the motion of the centre of pressure produced by various displacements of the given area.

Thus, if the area is rotated in its own plane about G, while G is fixed, the only variable in the values of ξ, η is a; and if this is eliminated from (3), (4), we have

$$\frac{\xi^2}{k_2^4} + \frac{\eta^2}{k_1^4} = \frac{1}{h^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is the locus described in the area by the centre of pressure—viz. an ellipse.

To find the locus described in this case by the centre of pressure with reference to fixed space, refer its position to the line GN and the horizontal line through G in the area as axes of x' and y', respectively. If (x', y') are the co-ordinates of I with reference to these axes, we have

$$\xi = x' \cos a - y' \sin a,$$
$$\eta = x' \sin a + y' \cos a.$$

Substituting the above values of ξ, η, and eliminating a, we have

$$\left(x' + \frac{k_1^2 + k_2^2}{2h}\right)^2 + y'^2 = \left(\frac{k_1^2 - k_2^2}{2h}\right)^2, \quad . \quad . \quad (6)$$

which shows that I describes a circle in fixed space, the centre of the circle being on the vertical through G.

Again, if the area is lowered into the liquid without rotation, h is the only variable in (3) and (4), by eliminating which we have

$$\frac{\eta}{\xi} = \frac{k_1^2}{k_2^2} \tan a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which shows that I describes a right line in the area; and it describes also a right line in space, for (7) gives a linear relation between x' and y'.

The equations (3) and (4) are of very great use in finding the centre of pressure on a plane area. They are both contained in the following rule: *the distance of I from either principal axis is equal to the k^2 about that axis divided by the intercept made on the other principal axis by the surface of the liquid.*

Cor. 1. If either principal axis at the centre of gravity of any plane area is vertical, the centre of pressure lies on that axis.

Cor. 2. If the momental ellipse of a plane area at G is a circle, the centre of pressure lies on the vertical through G.

The deeper the area in the liquid, the more nearly does the centre of pressure approach the centre of gravity; so that for a given direction of the line GA the centre of pressure will be farthest from G when the area touches the surface of the liquid (supposing the area to be completely immersed). Hence if the area be given all positions of complete immersion touching the surface, we get a curve locus of I which marks the extreme positions of this point. For all positions of complete immersion in which the area can be placed below the surface of the liquid the centre of pressure must lie within the area of this curve locus, which is called the *core* of the given area.

The core of any area is therefore found by drawing tangents to the area all round its contour and supposing these tangents to be successively the surface of the liquid in which the area is completely immersed, and for each of them determining the position of I; in other words, eliminating h and a from equations (3) and (4). The result will be a relation between ξ and η, which is the equation of the core.

EXAMPLES.

1. Find the position of the centre of water pressure on a circular area whose plane is vertical and whose centre is at a given depth.

Take as the principal axes of reference at G the vertical and horizontal diameters, and let h be the depth of G. Then in (3) and (4) we have $a = 0$ and $k_1^2 = k_2^2 = \frac{1}{4}r^2$, where r is the radius of the circle.

Hence
$$\xi = -\frac{r^2}{4h}, \quad \eta = 0,$$
so that I is on the vertical diameter at a depth $h + \frac{r^2}{4h}$ from the free surface.

If the area is just immersed, $h = r$, and the depth of I is
$$\tfrac{5}{4}r.$$

In the case of an elliptic area whose centre is at a depth h, and whose major axis makes an angle a with the vertical
$$\xi = -\frac{a^2}{4h}\cos a, \quad \eta = -\frac{b^2}{4h}\sin a,$$
where a and b are the semi-axes.

Hence the core of a circular area of radius r is a concentric circle of radius $\frac{r}{4}$. The core of an elliptic area is obtained by expressing h in terms of a, b and a. Now
$$h^2 = a^2\cos^2 a + b^2\sin^2 a,$$
since h is the perpendicular from the centre on a tangent. Eliminating h and a from the values of ξ and η, we have
$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = \frac{1}{16},$$

showing that the core is a concentric ellipse whose axes are one-fourth of those of the given ellipse.

2. If a plane area, wholly immersed in all positions, turns round any fixed point in its own plane, find the locus of the centre of pressure in the area.

Let P be the point about which it turns, α, β the co-ordinates of P with respect to the principal axes at G, ϖ the perpendicular from P on the surface of the liquid; then the equation of the locus referred to the principal axes at G is

$$(\alpha k_1^2 x + \beta k_2^2 y + k_1^2 k_2^2)^2 = \varpi^2 (k_1^4 x^2 + k_2^4 y^2),$$

which is a hyperbola, an ellipse, or a parabola according as

$$GP >, <, \text{ or } = \varpi.$$

If P is in the surface of the liquid, the locus is a right line.

3. A plane area which is a regular polygon of any number of sides is immersed in a liquid with one side in the surface; show that the depth of the centre of pressure below the centre of the area is

$$\frac{h}{6} + \frac{r^2}{12h},$$

where r is the radius of the circumscribing circle of the polygon, and h the perpendicular from the centre on a side.

(The momental ellipse at G is a circle, and the k^2 about any line at G in the area is $\frac{h^2}{6} + \frac{r^2}{12}$.)

4. If the plane area in the last question occupies any position of complete submergence, where is the centre of pressure?

Ans. On the vertical line through G at a depth

$$\frac{1}{p}\left(\frac{h^2}{6} + \frac{r^2}{12}\right)$$

below G, where p is the depth of G.

5. What is the core of an area in the shape of a regular polygon of n sides?

Ans. A regular polygon of n sides, each of length

$$\frac{1 + 2\sin^2 a}{12 \sin a} \cdot a,$$

where $2a$ is the length of a side and $2a$ the angle of the given polygon.

(Let AB be a side of the polygon lying in the surface. Then in this position I is on the perpendicular from G on AB at a depth $\frac{h}{6}+\frac{r^2}{12h}$ below G. Now let the area revolve about A; then, since we can take GA and its perpendicular at G as the principal axes of reference, Ex. 2 shows that the locus of the centre of pressure is a line, IH, perpendicular to GA at a depth, GH, which $=\frac{k^2}{r}$, below G; that is, at a depth $\frac{h^2}{6r}+\frac{r}{12}$ below G. The centre of pressure will describe this line until the next side, C, of the polygon comes into the surface, and at this moment the centre of pressure is at I' on the line IH such that $I'H = HI$, &c.)

6. What is the core of a parabolic area of latus rectum $4a$, bounded by a double ordinate of length $2c$?

Ans. A figure consisting of an arc of an ellipse and two right lines joining its extremities to a fixed point.

[Let the figure start from the position represented in Fig. 1, p. 1. In this position let I be the centre of pressure; then $OI = \frac{5}{7} OC = \frac{5c^2}{28a}$. Also $OG = \frac{3}{5} OC = \frac{3c^2}{20a}$. Taking GC and its perpendicular at G as the principal axes of x and y, we have $k_1^2 = \frac{c^2}{5}$, $k_2^2 = \frac{3c^4}{700a^2}$. Now let a tangent start from O and move up to B; then taking for each tangent the perpendicular from G and the angle a, and eliminating from the values of ξ and η in (3) and (4) of p. 10, we find the locus

$$\xi^2 + \frac{3c^2}{980a^2} \cdot \eta^2 - \frac{c^2}{35a} \cdot \xi = 0, \quad . \quad . \quad . \quad . \quad (a)$$

which is the locus until the tangent reaches B. When it reaches B, it must turn round B until it coincides with BC. During the process of turning, the centre of pressure (the revolving line being always imagined to coincide with the surface of the liquid) describes a right line, until the revolving line reaches BC, and then the centre of pressure, J, is on OC distant $\frac{4}{7}$ OC from O. The ordinate of the extremity of the elliptic arc in (a) is $\frac{c}{4}$, and the rectilinear part of the core joins this point to J. The core consists of two portions, one at one side and the other at the other side of GC.]

7. What is the core of a parallelogram $ABCD$?

Ans. A parallelogram whose sides are parallel to the diagonals AC, BD and equal to $\frac{1}{6}$ AC, $\frac{1}{6}$ BD.

8. What is the core of a triangular area ABC?

Ans. A triangular area IJK formed by the middle points of the bisectors of the sides drawn from the vertices.

9. Find the position of the centre of pressure of a semicircular area whose diameter is in the surface of water.

Result. If r is the radius of the circle, the centre of pressure is at a distance $\frac{3\pi}{16}r$ from the horizontal diameter. (The centre of gravity of a semicircle is $\frac{4r}{3\pi}$ from the centre.)

10. If the diameter is horizontal and at a depth h, find the depth of the centre of pressure.

Result. $\frac{r}{4} \cdot \frac{16h + 3\pi r}{4r + 3\pi h}$ below the horizontal diameter.

11. Find the position of the centre of pressure on a semicircular area whose bounding diameter is vertical with one extremity in the surface of water.

Result. Its distance from the vertical diameter is $\frac{4r}{3\pi}$, and its depth is $\frac{5}{4}r$.
(The point is on the vertical through the centre of gravity, G, of the area, since this is one of the principal axes at G.)

12. Find the position of the centre of pressure on a semicircular area completely immersed in water, the bounding diameter being inclined at an angle α to the horizon and having one

extremity in the surface of the water, and find the core of the area.

Result. Let G be the centre of gravity of the area, C the centre; draw GH perpendicular to GC; on CG produced through G take I such that $GI = \frac{9\pi^2 - 64}{48\pi} \cdot r$; on GH take $GJ = \frac{r}{4}$. The core consists of the line IJ and an elliptic arc.

Let us finally suppose the plane of the area to have any position whatever, which we shall define in the usual way by the Precession and Nutation angles θ, ϕ, ψ.

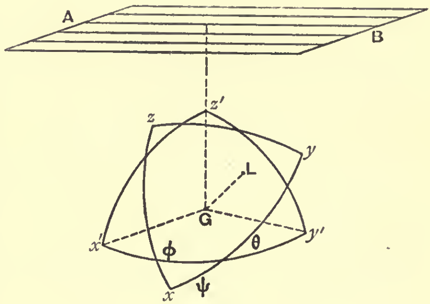

Fig. 6.

Take the vertical Gz' as axis of z', and any two rectangular horizontal lines, Gx', Gy', as axes of x' and y'. Let Gx, Gy be the principal axes at G in the plane of the given area, while Gz is the axis perpendicular to this plane.

Then, since the direction-cosines of Gz' with reference to the axes of x, y, z are

$$-\sin\psi\sin\theta, \quad \cos\psi\sin\theta, \quad \cos\theta,$$

the length of the perpendicular from any point $(x, y, 0)$ in the given area on the free surface AB is

$$h + x\sin\psi\sin\theta - y\cos\psi\sin\theta.$$

This multiplied by $w\,ds$ gives the pressure on the element of area, and the total moment of pressure about Gx is

$$-Ak_1^2 w\cos\psi\sin\theta.$$

Hence, as before,

$$\xi = \frac{k_2^2}{h} \sin\psi \sin\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\eta = -\frac{k_1^2}{h} \cos\psi \sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Suppose the area to be rotated, like a rigid body, round any line, GL, fixed in space, the direction-cosines of this line being l, m, n with reference to the space axes Gx', Gy', Gz'. Then the angles between GL and the principal axes Gx, Gy, Gz are all constant, and if we denote their cosines by λ, μ, ν, respectively, we find, by eliminating θ and ψ from (8) and (9) by means of the constants λ, μ, ν, the equation

$$\frac{\xi^2}{k_2^4} + \frac{\eta^2}{k_1^4} + \frac{1}{\nu^2}\left(\frac{\mu\xi}{k_2^2} + \frac{\mu\eta}{k_1^2} + \frac{n}{h}\right)^2 = \frac{1}{h^2}, \quad . \quad . \quad (10)$$

which gives the curve described in the area by the centre of pressure.

This agrees with (5) for the case in which the plane of the area is always kept vertical; for in this case

$$\theta = \frac{\pi}{2}, \ \psi = -a, \ \nu = 1, \ n = 0, \ \lambda = \mu = 0.$$

If the line GL is any one in the plane of the area $\nu = 0$, and the locus described by I in the area is a right line,

$$\frac{\lambda\xi}{k_2^2} + \frac{\mu\eta}{k_1^2} + \frac{n}{h} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

CHAPTER II

STABILITY OF A FLOATING BODY. THEORY OF THE METACENTRE

A THEORETICAL equilibrium position of a body floating freely in a liquid is any one in which the two conditions given in Art. 20 are fulfilled; but if the body is very slightly displaced from such a position and then left to itself, either of two things will happen—1, the body will return to its position of equilibrium; or 2, it will fall away still further from the position. If the first happens, the equilibrium is *stable*, and if the second, the equilibrium is *unstable*.

The conditions on which stability depends are the subject of this chapter.

53. Geometrical Theorem. In connexion with the question of the stability of floating bodies the following theorem is important.

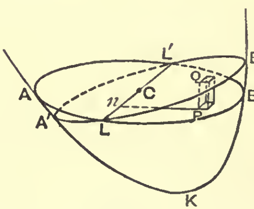

Fig. 7.

A volume AKB, Fig. 7, being cut off from a solid body by a plane section $ALBL'$, any other plane, $A'LB'L$, making a small angle with the first plane and cutting off an equal volume, $A'KB'$, must pass through the centroid (or 'centre of gravity'), C, of the area $ALBL'$.

For, at any point, P, in the plane section $ALBL'$ describe a small element of area, dS; let the perpendicular, Pn, from

P on the line, LL', of intersection of the two planes be denoted by x; let $\delta\theta$ be the angle between the two planes; and round the contour of dS draw perpendiculars to the plane of dS, these forming a prism which intersects the plane $A'LB'L'$ in a small area at Q. Then $\angle QnP = \delta\theta$, $QP = x\delta\theta$, and the volume of the small prism is very nearly $x\,dS\,.\,\delta\theta$. Hence the new volume $A'KB' =$ vol. $AKB + \delta\theta\int x\,dS$, the prisms in the wedge $L'LBB'$ being taken positively, while those in the wedge $L'LA'A$ are taken negatively. If the two volumes cut off are the same, we must have

$$\int x\,dS = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

the integration including all the elements of area of the plane section $ALBL'$. Now, by the theorem of mass-moments the left-hand side of (a) is $A\bar{x}$, where A is the area of the plane section, and $\bar{x}$ the distance of its centroid from the line LL'; hence $\bar{x} = 0$, i. e. the centroid of the area must lie on LL'.

A visible representation of this fact is obtained by holding in the hand a tumbler partly filled with water and imparting to it small and rapid oscillations which cause the surface of the water to oscillate from right to left; the planes of the successive surfaces of the water can then be seen to pass always through the centre of the horizontal section.

54. Small Displacements. Metacentre. Suppose a body, ACB, Fig. 8, floating in equilibrium in a homogeneous liquid to receive any small displacement; it is required to find whether the equilibrium is stable or unstable.

Every displacement can be regarded as consisting of two kinds of displacement—viz. a vertical displacement of translation, upwards or downwards, which diminishes or increases the volume of the displaced liquid, and a rotatory or side displacement which leaves the volume of the displaced liquid unaltered.

If the displacement is small, these component displacements can be treated separately, and it is evident that equilibrium for the first kind of displacement is stable.

We shall confine our attention, then, to displacements of rotation which leave the volume of the displaced liquid,

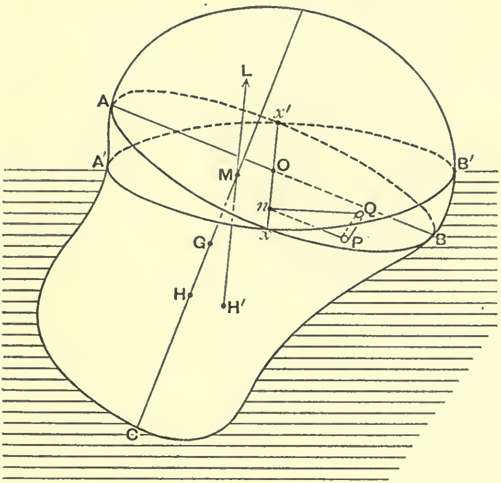

Fig. 8.

and therefore the magnitude of the force of buoyancy, unaltered.

Let G be the centre of gravity of the body, $Ax\ Bx'$ the section of flotation (i. e., the section of the body made by the free surface of the liquid) before displacement, H the centre of buoyancy (i. e. the centre of volume of the immersed portion ACB) before the displacement, and let this section of flotation be supposed to be marked on the surface of the body.

The body is represented as slightly displaced, the new section of flotation being $A'x B' x'$; the new centre of buoyancy (centre of volume $A'CB'$) is H', which must be somewhere very close to H, the centre of volume of ACB.

If the body, having been displaced to the new position, is then left to itself, it will be acted upon by two forces, viz. its weight, W, acting through G, and the force of buoyancy, L (which is equal to W, since the volume of the displaced liquid is constant), acting vertically upwards through H'. These forces form a couple.

Now it is clear that if the line $H'L$ cuts the line GH above G, in a point M, the body will be acted upon by a couple which tends to destroy the displacement; while, if M is below G, the moment of the couple, being in the sense of the displacement, will cause the body to fall farther from the position of equilibrium, which is therefore unstable.

It may happen, on account of the shape of the body and the position of the axis, $x'x$, of displacement, that the vertical line through H' does not intersect the old line GH of centres of gravity. At present we shall confine our attention to cases in which it does intersect GH, and subsequently we shall find the condition that such intersection shall take place.

Manifestly if G is below H, the equilibrium will be stable, and the consideration of this case may be dismissed. The case in which G is above H is very important inasmuch as it is the case of ships generally, and especially that of large ironclads, in which so much of the mass is in the upper portion.

If p is the length of the perpendicular from G on the line $H'L$, the moment

$$L \cdot p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

of the new force of buoyancy about G is called the *moment of stability.*

To calculate p, or the position of the point M—which point is called the *metacentre*—replace the actual force of buoyancy due to liquid $A'CB'$ by a force of buoyancy consisting of three components, viz.,

(*a*) an upward force due to liquid ACB,

(*b*) an upward force due to liquid $Bxx'B'$,

(*c*) a downward force due to liquid $Ax'xA'$.

The two sections of flotation intersect in the line xx', and since the volumes ACB and $A'CB'$ are equal, this line xx' must pass through the centroid, O, of the section of flotation (Art. 53).

Since the volume of the wedge $Bxx'B'$ = the volume of the wedge $Ax'xA'$, the forces (*b*) and (*c*), being the weights of these wedges of liquid, form a couple, each acting through the centre of gravity of the corresponding wedge; while force (*a*) is L acting up through H.

Also $L.p$ = the sum of the moments of these forces about the axis through G perpendicular to the plane of displacement. Now since the forces (*b*) and (*c*) form a couple, the sum of their moments about all parallel axes is the same, and hence the sum of their moments about the horizontal axis through G = the sum of their moments about xx', which latter we shall take. The wedges may be broken up into an indefinitely great number of slender prisms perpendicular either to the plane $A'xB'x'$ or to the plane $AxBx'$. Taking the latter mode, at any point P in the area $AxBx'$ describe an indefinitely small area dS, and round its contour erect perpendiculars which will cut off a small area at Q on the plane $A'xBx'$. Let AOB be the diameter at O perpendicular to $x'x$; take Ox and OB as axes of x and y; let the perpendicular Pn from P on $x'x$ be y, and let θ be the small angle, QnP, through which the body is displaced round $x'x$. Then the volume of the prism PQ is $\theta y\,dS$; its weight (reversed for buoyancy) acts at the middle point of PQ, and may be resolved into the components $\theta wy\,dS \cos\theta$ parallel to PQ and $\theta wy dS \sin\theta$ parallel to Pn; i.e. into components $\theta\,wydS$ and $\theta^2 wydS$ in these directions.

The moment of the latter, being of the second order in θ, may be neglected, while the moment of the former is

$$\theta\,wy^2 dS. \qquad (2)$$

By integrating (2) throughout both wedges, we obtain the sum of the moments of the forces (b) and (c), since they both give moments of the same sign about xx'.

Hence the moment of buoyancy due to the wedges is

$$\theta w\!\int\! y^2 dS, \qquad (3)$$

the integration extending all over the section $Ax\ Bx'$ of flotation.

If A is the area of this plane section, and k its radius of gyration about the axis xx' of displacement, (3) is

$$\theta w\,.\,Ak^2. \qquad (4)$$

We see, therefore, that the new forces of buoyancy—i. e., those acting in the displaced position—are equivalent to—

1° an upward thrust, Vw, acting at H, and

2° a counterclockwise couple, $\theta\,.\,Ak^2w$.

Now we know that a force F accompanied by a couple of moment M in the same plane compound into a force equal and parallel to F, at a perpendicular distance $\frac{M}{F}$ from the original force. Hence the resultant of 1° and 2° is an upward force equal to Vw acting in a line, $H'M$, distant

$\frac{\theta . Ak^2w}{Vw}$ from the point H; hence the perpendicular from H on $H'M$ is $\frac{Ak^2}{V} . \theta$; that is

$$HM = \frac{Ak^2}{V}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which determines the position of the metacentre, M.

For stability, therefore,

$$\frac{Ak^2}{V} > HG. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Displacements of constant volume may take place round any diameter of the section, $Ax\,Bx'$, of flotation provided that the diameter passes through the centroid of this section (Art. 53); and since for all such displacements both A and V are constant, equation (5) shows that the metacentre will be highest when the displacement takes place round that diameter about which the moment of inertia of the section of flotation is greatest, and lowest if it takes place round the diameter about which the moment of inertia is least. These two diameters are the *principal axes* of the section of flotation at its centroid. If k_2 and k_1 are the greatest and least radii of gyration of the section of flotation about its principal axes, and M_2, M_1, the corresponding metacentres for displacements round them,

$$HM_2 = \frac{Ak_2^2}{V}, \quad HM_1 = \frac{Ak_1^2}{V}. \quad . \quad . \quad . \quad . \quad (7)$$

The equilibrium will, then, be least stable when the displacement takes place round the diameter of least moment of inertia, which in the case of a ship is the line from stem to stern.

Since $Vw = W$, (5) can be written

$$HM = \frac{Awk^2}{W}. \quad \quad (8)$$

55. Experimental determination of Metacentre. The height of the metacentre above the centre of gravity of a ship can be found experimentally by means of a plumb-line and a movable mass on the deck. Suppose one end of a long string fastened to the top of one of the masts and let a heavy particle hang from the other end of the string. Now if a considerable mass, P, be shifted from one side of the deck to the other, the ship will be tilted through a small angle which can be measured by means of the pendulum if the bob of the pendulum moves in front of a vertical sheet of paper on which the amount of displacement of the bob can be marked. If l is the length of the string and s the distance traversed on the paper by the bob while the mass P is shifted across the deck, $\frac{s}{l}$, is the circular measure of the whole angle of deflection of the ship.

Let G be the centre of gravity of the ship, W the weight of the ship and movable mass together, $2b$ the breadth of the deck, a the perpendicular from G on the plane of the deck, and 2θ the whole angle, $\frac{s}{l}$, of deflection. Then, on account of the symmetry of the ship, we can in Fig. 8 take the line HG as passing through O.

Let the mass P be at B, and take moments of the forces acting about G; then

$$W.GM.\theta = P(b + a\theta),$$

$$\therefore \quad GM = \frac{P}{W}\left(\frac{b}{\theta} + a\right).$$

The value of a is usually much smaller than $\frac{b}{\theta}$, so that, with sufficient accuracy, we have

$$GM = \frac{P}{W} \cdot \frac{b}{\theta},$$

where $\theta = \frac{2s}{l}$.

Thus, in a ship of 10,000 tons the breadth of whose deck is 40 feet, if a mass of 50 tons moved from one side to the other causes the bob of a plumb-line 20 feet long to move over 10 inches, the metacentric height is about $4\frac{4}{5}$ feet.

The metacentric heights of large war vessels vary from about $2\frac{1}{2}$ feet to 6 feet.

Examples of the Metacentre.

1. A uniform rectangular block, of specific weight w', floats, with one of its edges vertical, in a liquid of specific weight w; find the relation between its linear dimensions so that the equilibrium shall be stable.

Let $2a$, $2b$ be the lengths of the horizontal edges, and $2c$ the length of the vertical edge, and let $b < a$. Then the equilibrium is most unsafe when a displacement is made round the longest diameter of the section of flotation. If x is the length of the vertical edge immersed,

$$x = 2c\frac{w'}{w}.$$

and therefore

$$HG = c\left(1 - \frac{w'}{w}\right),$$

Also $k^2 = \frac{1}{3}b^2$ round the axis of most dangerous displacement, and $V = \frac{W}{w}$, where $W = 8abcw'$ = weight of body. Hence

$$HM = \frac{b^2 w}{6cw'},$$

so that for stability $\frac{b^2 w}{6 c w'} > c\left(1 - \frac{w'}{w}\right)$, that is

$$\frac{b}{c} > \sqrt{6\frac{w'}{w}\left(1 - \frac{w'}{w}\right)}.$$

2. If the floating body is a solid cylinder, floating with its axis vertical, find the condition for stability.

Result. If r is the radius of the base and h the height,

$$\frac{r}{h} > \sqrt{2\frac{w'}{w}\left(1 - \frac{w'}{w}\right)}.$$

3. If the floating body is a solid cone, floating with its axis vertical and vertex downwards, find the condition for stability.

Result. If r is the radius of the base and h the height,

$$\frac{r}{h} > \sqrt{\left(\frac{w}{w'}\right)^{\frac{1}{3}} - 1}.$$

4. If the floating body is a solid isosceles prism whose base is uppermost, find the condition for stability.

Result. If $2b$ is the length of the shorter side of the base and h the height of the prism,

$$\frac{b}{h} > \sqrt{\left(\frac{w}{w'}\right)^{\frac{1}{2}} - 1}.$$

5. If the cone in example 3 floats with its vertex uppermost, find the condition for stability.

Result. $$\frac{r}{h} > \sqrt{\left(\frac{w}{w - w'}\right)^{\frac{1}{3}} - 1}.$$

6. A solid homogeneous prism whose cross-section is an isosceles triangle of height 12 feet and base 12 feet floats with its vertex downward in water. If on its base is constructed another isosceles prism of the same substance, find the height of this prism when the equilibrium becomes unstable for lateral displacements, the specific gravity of the prism being $\frac{2}{3}$.

Result. 6 feet.

7. A homogeneous paraboloid of revolution of specific weight w', height h, and latus rectum $4a$ floats, vertex down, in a liquid of specific weight w; show that, whatever be the height, $HM = 2a$, and that for stability

$$a > \frac{h}{3}\left\{1 - \left(\frac{w'}{w}\right)^{\frac{1}{2}}\right\}.$$

8. Show that if the paraboloid floats with its vertex uppermost the condition for stability is

$$a > \frac{h}{3}\left\{\frac{w}{w'} - \frac{\sqrt{w(w-w')}}{w}\right\}.$$

9. Prove that if a homogeneous right circular cone whose vertical angle is 60° floats in any liquid with its vertex downward the metacentre will lie in the plane of flotation.

10. A solid homogeneous prism whose section perpendicular to its edge consists of the curve whose equation is $c^{n-1}y = x^n$ and its reflexion in the axis of x (which axis is perpendicular to the edge) floats in a liquid with the edge submerged. If h is the height of the prism, find the condition for stability.

Result. $$\left\{1 + \tfrac{1}{3}(n+2)\frac{h^{2n-2}}{c^{2n-2}}\left(\frac{w'}{w}\right)^{\frac{2n-2}{n+1}}\right\}\left(\frac{w'}{w}\right)^{\frac{1}{n+1}} > 1,$$

where w, w' are the specific weights of liquid and prism.

11. A half cylinder, obtained by cutting a solid homogeneous cylinder by a plane through its axis, floats, with face of section vertical, in a liquid; show that it will be stable if

$$\frac{r^2}{h^2} > \frac{50}{7}\frac{w'}{w}\left(1 - \frac{w'}{w}\right).$$

12. A cylinder whose cross-section is any curve, ADB (Fig. 9), symmetrical about an axis DO in its plane, floats with its axis horizontal in a liquid; prove that if the metacentre coincides with the centre of gravity (or if the equilibrium is apparently neutral), the equilibrium is stable or unstable according as $\rho \sin \epsilon >$ or $< OB$, where ρ is the radius of curvature of the curve at B, ϵ is the angle between OB and the tangent at B, and AOB is the original water-line.

Let the figure represent the body slightly displaced; let the equation of the curve referred to O as origin and OB as initial line be

$$\rho = f(\phi), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\rho = OB'$, $\phi = BOB'$. Let C be the point of intersection of the new water-line $A'B'$ and AB; $OC = c$, $CB' = r$, $BCB' = \theta$.

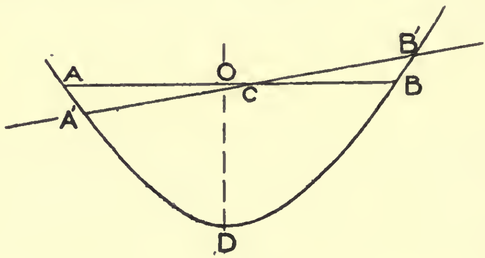

Fig. 9.

We shall find the equation of the curve referred to C as pole, and retain terms in θ as far as θ^2 and assume that c is a small quantity compared with any value of r.

Then $\rho^2 = r^2 + 2cr\left(1 - \frac{\theta^2}{2}\right) + c^2 = (r+c)^2$, by supposition

$$\therefore \ \rho = r + c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again,

$$\rho \sin\phi = r \sin\theta$$

$$\therefore \ \phi = \theta\left(1 - \frac{c}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now $\rho = f(0) + \phi f'(0) + \frac{\phi^2}{2} f''(0)$; therefore substituting for ρ and ϕ we have

$$r^2 - (f - c + \theta f' + \tfrac{1}{2}\theta^2 f'')\, r + c\theta f' = 0 \quad . \quad . \quad . \quad (4)$$

where f is used for $f(0)$, &c.

This gives

$$r = f - c + \theta f' + \tfrac{1}{2}\theta^2 f'' - c\theta\frac{f'}{f} \quad . \quad . \quad . \quad . \quad (5)$$

Now the buoyancy due to the wedge represented by the area BCB' may be found by breaking the area into triangles such as PCQ (Fig. 10), and the buoyancy represented by this triangle acts at g, its centre of gravity, vertically upwards, its moment about C being

$$\tfrac{1}{3}r^3 d\theta \,.\, \cos(\beta-\theta),$$

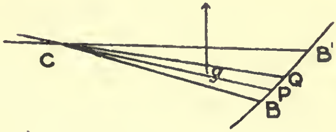

Fig. 10.

if we now use β to denote the whole angle BCB', the angle PCB being θ.

The real moment of buoyancy about the axis through C parallel to the axis of the cylinder is the above multiplied by lw, where l is the length of the axis and w the weight per unit volume of the liquid. We shall omit lw for the present.

Hence the moment for the wedge BCB' is

$$\tfrac{1}{3}\int_0^\beta r^3\left[1-\tfrac{1}{2}(\beta-\theta)^2\right]d\theta \quad . \quad . \quad . \quad . \quad (6)$$

Now (5) gives

$$r^3 = f^3 - 3cf^2 + 3\theta f^2 f' - 9c\theta ff' + \tfrac{3}{2}\theta^2(f^2f'' + 2ff'^2) + 3c^2 f,$$

so that (6) becomes

$$\tfrac{1}{3}f^3\beta - cf^2\beta + \tfrac{1}{2}f^2 f'\beta^2 - \tfrac{3}{2}cff'\beta^2 + \tfrac{1}{6}(f^2f'' + 2ff'^2)\beta^3$$
$$+ c^2 f\beta - \tfrac{1}{18}f^3\beta^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now if

$$OB = a, \text{ we have } f(0) = a, \; f'(0) = a\cot\epsilon,$$

$$f''(0) = \frac{a}{\sin^2\epsilon}\left(1+\cos^2\epsilon - \frac{a}{\rho\sin\epsilon}\right).$$

[The reason of the last is that in any curve $r = f(\theta)$, we have

$$f'(\theta) = r\cot\phi;\; f''(\theta) = r\cot^2\phi - \frac{r}{\sin^2\phi}\cdot\frac{d\phi}{d\theta}, \text{ and}$$

$$\frac{d\phi}{d\theta} = \frac{d\psi}{d\theta} - 1 = \frac{r}{\rho\sin\phi} - 1,$$

where ϕ is the angle between radius vector and tangent, and $d\psi$ = angle between two consecutive tangents.]

For the wedge ACA' we have c replaced by $-c$ and ϵ replaced by $\pi-\epsilon$, so that f, f'' remain unaltered while c and f' change sign.

Hence, adding the righting moment of the wedge ACA' to (7), the second and third terms disappear, and the whole moment is lw multiplied by

$$(\tfrac{2}{3}f^3+2c^2f)\beta-3ff'c\beta^2+\tfrac{1}{3}(2ff'^2+f^2f''-\tfrac{1}{3}f^3)\beta^3 \quad . \quad (8)$$

If we assume the volume of immersion constant, the areas BCB' and ACA' are equal, *i.e.*, $\int r^2d\theta$ is the same for both. This gives in the same way

$$(f^2-2cf+c^2)\beta+(ff'-2cf')\beta^2+\tfrac{1}{3}(f'^2+ff'')\beta^3$$

the same for both, and therefore $c=\frac{1}{2}f'\beta$.

Substituting this in (8), we have

$$\tfrac{2}{3}f^3\beta+\tfrac{1}{9}f(3ff''-3f'^2-f^2)\beta^3,$$

so that if W is the weight of the cylinder and K its radius of gyration about an axis through G parallel to the axis of the cylinder, the equation of motion is

$$\frac{WK^2}{g}\cdot\frac{d^2\beta}{dt^2}$$
$$=Vw\,.\,GH\sin\beta-lw\left[\tfrac{2}{3}f^3\beta+\tfrac{1}{9}f(3ff''-3f'^2-f^2)\beta^3\right].$$

Putting $\sin\beta=\beta-\frac{1}{6}\beta^3$, and assuming that the term in β vanishes, we have $Vw\,.\,GH=\frac{2}{3}lwf^3$, and then

$$\frac{WK^2}{g}\frac{d^2\beta}{dt^2}=-\tfrac{1}{3}flw(ff''-f'^2)\beta^3,$$

which shows that the motion will be oscillatory if $ff''>f'^2$, and this is the same as the result at first announced.

13. A homogeneous circular cylinder floats with its axis vertical in a liquid; show that if the equilibrium is apparently neutral it is stable for all displacements, however great.

Let Oxy (Fig. 11) be the original section of flotation and DOx the new section intersecting the first in the diameter xx'. We have to calculate the moment of the couple due to the buoyancy of the wedge Dy and the downward action due to $D'y'$. Take a section efk of the wedge Dy made by a plane at a height ky, or z, above Oxy, and take a strip pq of this section.

Now if m denotes the tangent of the angle DOy, the co-ordinates of f are $\frac{\sqrt{m^2a^2-z^2}}{m}$, $\frac{z}{m}$, z. The centre of the circle ekf is on the axis Oz at C. If the angle kCq is ϕ the area of the strip pq is

$$2a^2 \sin^2\phi \, d\phi,$$

and if we consider the plate formed by the section ekf and an infinitely close parallel one, the distance between them being dz, we have the elementary volume

$$2a^2 \sin^2\phi \, d\phi \, dz$$

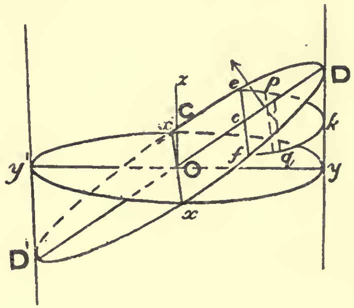

Fig. 11.

of liquid whose buoyancy acts perpendicularly to DD' and is represented by the arrow. The moment of this about xx' is

$$2wa^2 \sin^2\phi \, d\phi \, dz \left[Oc + (a\cos\phi - Cc)\cos\theta\right],$$

where $\theta = \angle DOy = \tan^{-1} m$. This moment is

$$2wa^2 \sin^2\phi \,(z\sin\theta + a\cos\theta\cos\phi)\, d\phi \, dz.$$

Integrating this for the section ekf, we have

$$wa^2 \,.\, z\sin\theta\,(a - \sin a\cos a) + \tfrac{2}{3} wa^3 \sin^3 a \cos\theta,$$

where $a = eCc$. Now $z = ma\cos a$, $\therefore$ the moment for the whole wedge is

$$\tfrac{1}{4} wa^4 \sin\theta \tan^2\theta \int_0^{\frac{\pi}{2}} (2a\sin 2a - \sin^2 2a)\, da$$
$$+ \tfrac{2}{3} wa^4 \sin\theta \int_0^{\frac{\pi}{2}} \sin^4 a \, da$$

or

$$\frac{\pi}{16} wa^4 \sin\theta\,(2 + \tan^2\theta)$$

and the moment of the couple of buoyancy due to the two wedges is double this, so that the equation of rotation about G is, as in the previous question

$$\frac{WK^2}{g}\frac{d^2\theta}{dt^2} = Vw \,.\, HG\sin\theta - \tfrac{1}{8}\pi a^4 w \sin\theta\,(2 + \tan^2\theta).$$

If the equilibrium is neutral, $Vw = \frac{1}{4}\pi a^4 w$, and we have

$$\frac{WK^2}{g}\frac{d^2\theta}{dt^2} = -\tfrac{1}{8}\pi a^4 w \sin\theta \tan^2\theta,$$

which shows that the motion is always oscillatory.

The same result holds for an elliptic cylinder, a^4 being replaced by ab^3 or a^3b according as the displacement is made round the axis a or the axis b.

We have assumed that H', Fig. 8, lies in the plane of displacement, and we can easily see that this will not be the case unless the axis, $x'x$, of displacement is a principal axis of the section, $AxBx'$, of flotation. For, if we seek the co-ordinates of H' (which is the centre of volume of the new volume $A'CB'$) we may regard, as before, the volume $A'CB'$ as 'resolved' into the original volume ACB, the positive wedge $B'xBx'$, and the negative wedge $A'xx'A$. Hence if x is the distance of the point P from the line OB and ξ the distance of H' from the vertical plane containing OB, we have

$$V.\,\xi = \theta \int xy\,dS,$$

since the volume of the prism PQ is $\theta y dS$, and its volume-moment about OB is $\theta xy dS$, the integration extending all over the area $AxBx'$.

This shows that $\xi = 0$ only when Ox and OB are principal axes at O. In the case of a square or circular section of flotation, every axis through O is a principal axis, and hence H' always lies in the plane of displacement.

In general, therefore, a small angular displacement round a diameter of the section of flotation produces a moment of the forces not only round this axis but also round the perpendicular axis in the plane of flotation, the effect of which would be to produce small oscillations of the body about this axis.

The question of stability, however, is not affected by this consideration, since any small angular displacement, θ, round an axis $x'x$ could be resolved into two separate small angular displacements

$$\theta \cos a \quad \text{and} \quad \theta \sin a$$

round the two *principal* axes at O, where a is the angle made by $x'x$ with one of these principal axes; and if the equilibrium were stable for small displacements round the most dangerous of the principal axes, it would be so for the given displacement round $x'x$.

Let us investigate the lines of action of the force of buoyancy when the displacements are made round various diameters of the section of flotation. In the position of the body previous to displacement the line OB (Fig. 8) was horizontal. At O draw a downward axis perpendicular to Ox and OB, and let this be the line Oz' in Fig. 12, in which Oy represents OB' and Oy' represents OB, the displacement being round Ox, and θ, the angular displacement being yOy' or zOz'.

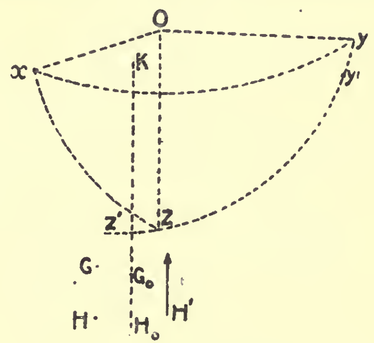

Fig. 12.

Let G_0 and H_0 be the original positions of G and H, in the same vertical line, and let G and H be their positions after displacement, the new centre of buoyancy, as in Fig. 8 being H'. The line H_0G_0 cuts the original plane of flotation in K.

With reference to Ox, Oy, Oz let the co-ordinates of G_0 be $\bar{x}_0, \bar{y}_0, \bar{z}_0$, and let those of H_0 be ξ_0, η_0, ζ_0. Then

$$\xi_0 = \bar{x}_0 \text{ and } \eta_0 = \bar{y}_0.$$

Now the direction-cosines of the new axes, Ox', Oy', Oz' with reference to Ox, Oy, Oz are represented in the diagram

	x	y	z
x'	1	0	0
y'	0	1	θ
z'	0	$-\theta$	1

so far as the first power of θ is concerned.

Hence with reference to Ox, Oy, Oz the co-ordinates of

$$G \text{ are } \xi_0,\ \eta_0 - \theta \bar{z}_0,\ \ \bar{z}_0 + \theta\eta_0$$

and of

$$H \ \ ,, \ \ \xi_0, \eta_0 - \theta\zeta_0,\ \zeta_0 + \theta\eta_0.$$

Hence the equations of the line HG with reference to Ox, Oy, Oz are

$$x - \xi_0 = 0,\ \frac{y - \eta_0}{-\theta} = z - \theta\eta_0 \ . \quad . \quad . \quad . \quad (1)$$

Now let ζ', η', ξ' be the co-ordinates of H' (which is the centre of mass of the volume $A'CB'$) with reference to Ox', Oy', Oz'. Then take the equations of mass-moments with reference to the planes $y'z'$, $z'x'$, $x'y'$. In each case we can consider the volume $A'CB'$ as resolved into ACB, the positive wedge $B'B$, and the negative wedge AA'. The volume of the small prism PQ is $\theta y' dS$, and its mass-moment with reference to the plane $y'z'$ is $\theta x'y'dS$; so that the total mass-moment of the wedges about this plane is $\theta \int x'y'dS$, that is, the product of inertia of the area of flotation about the axes Ox', Oy' multiplied by θ. Denote this product by $A\varpi$, where A is the area of flotation. The mass-moment of the volume ACB with reference to the plane $y'z'$ is, of course, $V\xi_0$ or $V\bar{x}_0$. Hence

$$V\xi' = V\xi_0 + \theta A\varpi.$$

With reference to the plane $z'x'$ the mass-moment of the wedges is θAk^2; and the moment of ACB is $V\eta_0$; therefore

$$V\eta' = V\eta_0 + \theta Ak^2.$$

With reference to the plane $x'y'$, the wedge-moment is of the order θ^2, and the moment of ACB is $V\zeta_0$; therefore

$$V\zeta' = V\zeta_0.$$

Hence we have

$$\xi' = \xi_0 + \theta\frac{A\varpi}{V},$$

$$\eta' = \eta_0 + \theta\frac{Ak^2}{V},$$

$$\zeta' = \zeta_0;$$

and if ξ, η, ζ are the co-ordinates of H' with reference to Ox, Oy, Oz,

$$\xi = \xi_0 + \theta\frac{A\varpi}{V},$$

$$\eta = \eta_0 + \theta\left(\frac{Ak^2}{V} - \zeta_0\right),$$

$$\zeta = \zeta_0 + \theta\eta_0.$$

Hence the equations of the vertical line through H' with reference to Ox, Oy, Oz are

$$\left.\begin{aligned} x &= \xi_0 + \theta\frac{A\varpi}{V}, \\ y &= \eta_0 + \theta\left(\frac{Ak^2}{V} - \zeta_0\right); \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

and if this intersects the line given by (1), we must have $\varpi = 0$, that is, the product of inertia of the area of flotation about Ox', Oy' must be zero, as shown before; and then

$$z = \zeta_0 - \frac{Ak^2}{V}$$

for the point of intersection. Now $\zeta_0 - z$ is HM in Fig. 8, and this equation gives the result

$$HM = \frac{Ak^2}{V}.$$

Consider displacements round all diameters, such as Ox, of the section of flotation. Let Ox make the angle ϕ with one of the *principal* axes of this section at O; let k_1 be the radius of gyration of the section about this principal axis, and let k_2 be the other principal radius of gyration.

Then

$$\varpi = \tfrac{1}{2}(k_1^2 - k_2^2)\sin 2\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$k^2 = \tfrac{1}{2}(k_1^2 + k_2^2) + \tfrac{1}{2}(k_1^2 - k_2^2)\cos 2\phi \quad . \quad . \quad (4)$$

Observe that in the above equations ξ_0, η_0, or $\bar{x}_0$, $\bar{y}_0$, are the co-ordinates of G_0 referred to the original positions of the axes Ox', Oy' in the plane of flotation, the contemplated displacement, θ, taking place about the axis Ox', which we now vary. Hence we must express $\bar{x}_0$, $\bar{y}_0$ in terms of a, β, the co-ordinates of G_0, or K, with reference to the *principal* axes of the area of flotation at O.

Thus we must put

$$\xi_0 = a\cos\phi + \beta\sin\phi\,; \quad \eta_0 = \beta\cos\phi - a\sin\phi\,;$$

also the equations of the vertical line through H' with reference to Ox', Oy', Oz' are obtained by putting

$$x = x', \; y = y' - \theta z'$$

in (2), so that we obtain

$$x' = \xi_0 + \theta\frac{A\varpi}{V},$$

$$y' - \theta z' = \eta_0 + \theta\left(\frac{Ak^2}{V} - \zeta_0\right).$$

Let this line cut the plane $x'y'$ in a point whose co-ordinates are X', Y', 0;

$$\therefore\ X' = \xi_0 + \theta\frac{A\varpi}{V},$$

$$Y' = \eta_0 + \theta\left(\frac{Ak^2}{V} - \zeta_0\right).$$

Expressing X', Y' in terms of X, Y, the co-ordinates of the point of intersection with reference to the principal axes at O, we have

$$(X-\alpha)\cos\phi + (Y-\beta)\sin\phi = \frac{\theta A}{2V}(k_1^2 - k_2^2)\sin 2\phi,$$

$$-(X-\alpha)\sin\phi + (Y-\beta)\cos\phi$$

$$= \frac{\theta A}{2V}\left[k_1^2 + k_2^2 + (k_1^2 - k_2^2)\cos 2\phi\right] - \theta\zeta_0,$$

which give

$$X - \alpha = -\theta\left(\frac{Ak_2^2}{V} - \zeta_0\right)\sin\phi,$$

$$Y - \beta = \theta\left(\frac{Ak_1^2}{V} - \zeta_0\right)\cos\phi,$$

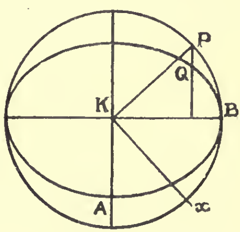

Fig. 13.

showing that the points in which the plane of flotation is intersected by the lines of action of the forces of buoyancy lie on a small ellipse having the point α, β for centre, its semi-axes being proportional to $\frac{Ak_2^2}{V} - \zeta_0$ and $\frac{Ak_1^2}{V} - \zeta_0$, which are the distances from the plane of flotation of the metacentres corresponding to small displacements round the principal axes of the section of flotation.

The point, Q, in which the section of flotation is intersected by the line of action of the force of buoyancy when the displacement is made round a diameter is thus found: let $k_1 > k_2$; through the point K of Fig. 13 draw axes

KA, KB, Fig. 13, parallel to the principal axes at O of the area of flotation; about K describe the ellipse whose semi-axes KA, KB are $\theta\left(\frac{Ak_2^2}{V}-\zeta_0\right)$ and $\theta\left(\frac{Ak_1^2}{V}-\zeta_0\right)$; describe the circle about K with radius KB; let Kx be parallel to the diameter Ox (Fig. 12) of displacement; draw KP perpendicular to Kx, meeting the circle in P; then a perpendicular from P to KB meets the ellipse in Q.

56. Constrained Displacements. Suppose now that the displacement takes place about an axis which is not a diameter of the area of flotation at its centre of gravity. Then the volume of the displaced liquid is not constant.

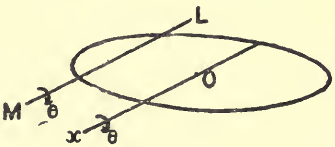

Fig. 14.

Suppose the body to receive a small angular displacement about an axis LM (Fig. 14) in the section of flotation but not passing through its centre of gravity, O. Now the second position of the body can be produced by combining two motions: (a) a rotation θ about a diameter Ox parallel to LM, and (b) a motion of translation of the whole body perpendicular to the plane of the two parallel axes, this motion being of the amount $p\,.\,\theta$, where p is the perpendicular distance between LM and Ox. This motion is a vertical one. The actual effect of buoyancy is obtained by superposing the buoyancy forces due to these two displacements. But, by what has gone before, the displacement (a) gives a force of buoyancy equal to Vw acting at H' (Fig. 8

or Fig. 12), where V is the original volume of the immersed part of the body; and if the axis LM is not fixed in the body, but released when the displacement has taken place, we must consider the moments of this force about horizontal axes drawn at G parallel and perpendicular to Ox — i.e., parallel to Ox and Oy in Fig. 12. These moments are

$$Vw(\eta-\bar{y}) \text{ and } -Vw(\xi-\bar{x}),$$

where $\bar{y}$ and $\bar{x}$ are the co-ordinates of G given on p. 36; that is

$$\bar{y}=\eta_0-\theta\bar{z}_0,\ \bar{x}=\xi_0.$$

Hence these moments are

$$\theta Vw\left(\frac{Ak^2}{V}-\zeta_0+\bar{z}_0\right) \text{ and } -\theta w A\varpi.$$

If, as supposed in Fig. 8, there is a metacentre, i.e., if Ox is a principal axis of the area of flotation, the second moment vanishes and the first is

$$\theta w\,(Ak^2-V.\,HG),$$

as in p. 24.

Now the motion (b) results simply in adding the volume $A\,.\,p\,.\,\theta$ to the volume of displaced liquid; and this gives a force of buoyancy equal to

$$\theta Apw$$

acting upwards through O, and its moments about the axes at G parallel to Ox and Oy are, neglecting θ^2,

$$\theta Apw\,.\,\eta_0,\ -\theta Apw\,.\,\xi_0.$$

These must be added to the previous moments.

Thus the total moment tending to diminish θ is

$$\theta w\left[A(k^2+p\eta_0)-V.\,HG\right],$$

and the stability depends on the sign of this moment. If the moment is positive, θ oscillates between small limits.

Suppose now that the axis, LM, of displacement (Fig. 15) has any horizontal position; through O, the centroid of the area of flotation, draw Ox parallel to LM; let p be the perpendicular distance between the axes, and ϵ the angle which the plane of the axes makes with the section of flotation. As before, resolve the rotation θ about LM into θ about Ox and a motion of translation $p\theta$ perpendicular to the plane of LM and Ox. Then the buoyancy due to the rotation about Ox is given in the previous discussion, and that due to the translation can be found by resolving the displacement $p\theta$ into a vertical motion

$$p\,\theta \cos \epsilon$$

and a horizontal motion

$$p\,\theta \sin \epsilon.$$

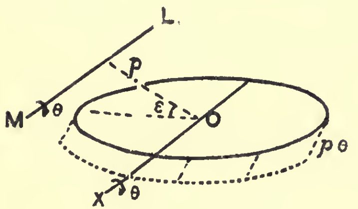

Fig. 15.

We assume, for the present, that, as in the last case, the axis LM is not rigidly attached to the body, and that the body is left free after the rotation θ about LM has been produced. Then, in considering the restoring moment about G, we may neglect the horizontal displacement $p\,\theta \sin \epsilon$ which does not alter the volume of the displaced liquid.

The result is then the same as that just found, $p\,\theta \cos \epsilon$ replacing $p\,\theta$.

If, however, in this and in the previous case LM is a fixed axis about which the body is constrained to continue turning, we must take the restoring moment about LM instead of about G.

In the previous case (Fig. 14) the forces acting on the body in the displaced position, other than the reaction of the fixed axis LM, are the weight, W, of the body acting downwards at G, the buoyancy Vw acting upwards at H' (Fig. 12), and the buoyancy $Ap\,\theta\,w$ acting upwards through O. Thus the total restoring moment about LM is

$$Vw\left[\eta_0+\theta\left(\frac{Ak^2}{V}-\zeta_0\right)+p\right]-W(\bar{y}_0-\theta\bar{z}_0+p)+\theta Ap^2w,$$

or

$$\theta\left[Aw(k^2+p^2)-Vw\,.\,\zeta_0+W\bar{z}_0\right],$$

since the term independent of θ vanishes, and the original co-ordinates, $\bar{x}_0$, $\bar{y}_0$, of G are not necessarily the same as ξ_0, η_0, the points G_0 and H_0 (Fig. 12) not necessarily lying on the same vertical line.

When LM has the position shown in Fig. 15, we superpose the following displacements: a rotation θ about Ox, a vertical downward translation $\theta\,p\cos\epsilon$ of the body, and a horizontal translation $\theta\,p\sin\epsilon$ of the body towards the left. The first gives the positions of G and H' shown in Fig. 12, at which act the force W downwards and the force Vw upwards; the second gives the additional force of buoyancy $\theta\,Awp\cos\epsilon$ acting upwards at O; and the last brings the points of action of application of these forces nearer, horizontally, to LM by the amount $\theta\,p\sin\epsilon$. Hence the total moment round LM against the displacement is

$$Vw\left[\eta_0+\theta\left(\frac{Ak^2}{V}-\zeta_0\right)+p\cos\epsilon-\theta p\sin\epsilon\right]$$
$$-W(\bar{y}_0-\theta\bar{z}_0+p\cos\epsilon-\theta\,p\sin\epsilon)+\theta\,Awp^2\cos^2\epsilon.$$

Now, on account of equilibrium when θ was zero, the term in this independent of θ vanishes, so that the restoring moment is

$$\theta\left\{Vw\left(\frac{Ak^2}{V}-\zeta_0-p\sin\epsilon\right)+W(\bar{z}_0+p\sin\epsilon)+Awp^2\cos^2\epsilon\right\}\quad(a)$$

Examples.

1. A homogeneous solid circular cylinder whose height is 6 feet and radius 2 feet is movable round a fixed horizontal diameter of one of its bases, the other base and a certain length of the cylinder being immersed in water. If the specific gravity of the cylinder is $\frac{1}{2}$, find the greatest length that can be immersed consistently with stability for small displacements round the axis.

Result. 2 feet.

2. If in the last case the height of the cylinder is h, the radius r, the specific weight w', and the specific weight of the liquid is w, find the condition of stability.

Result. The length unimmersed must be not less than

$$\sqrt{\left(1 - \frac{w'}{w}\right) h^2 - \frac{r^2}{2}}.$$

3. A homogeneous right circular cone freely movable round a fixed horizontal axis coinciding with a diameter of its base rests with a certain length of its axis immersed in water; find the greatest length immersed consistently with stability.

Result. If h and r are the height and radius of base, w' and w the specific weights of the solid and liquid, the length, x, of axis under the water is not greater than the value given by the equation

$$3(r^2 + h^2)x^4 - 4h^3x^3 + \frac{w'}{w} h^6 = 0.$$

4. If in the last the cone is movable round its vertex which is fixed above the liquid, find the least length of the axis unimmersed.

Result. The length x is given by the equation

$$3(h^2 + r^2)x^4 - 4h^5x + 3h^6\frac{w'}{w} = 0.$$

5. A homogeneous isosceles triangular prism, whose section perpendicular to its edge is a triangle whose height is h and base $2c$, is movable round a horizontal axis parallel to its edge and passing through the middle point of the base is partially

immersed in a liquid; find the greatest depth of immersion of its vertex consistently with stability.

Result. The greatest depth, x, of the vertex is given by the equation

$$2(c^2+h^2)x^3-3h^3x^2+\frac{w'}{w}h^5=0,$$

where w, w' are the specific weights of the liquid and solid.

6. A solid homogeneous isosceles prism whose section perpendicular to its edge is the triangle ABC is placed with its edge downwards in a liquid and allowed to assume its free equilibrium position, AB being horizontal. An axis parallel to the edge is then fixed through A; find the condition for stability.

Result. If $\frac{w}{w'}=n^2$, where w, w' are the specific weights of the liquid and solid, and h, $2b$ are the height and base AB of the triangle ABC,

$$\frac{b^2}{h^2}>2\frac{n-1}{3n^2+1}.$$

Compare this with the result in example 4, p. 28, and we see that for stability in the perfectly free condition of floating the ratio $b:h$ is greater than that required when the body is constrained—as is evident *a priori*.

7. A homogeneous solid circular cylinder is placed with its axis vertical in a liquid and allowed to take the free equilibrium position. If it is unstable in this position, can it be rendered stable, (a) by fixing a diameter of its upper base, (b) by fixing a horizontal tangent of the rim of its upper base?

Result. (a), no; (b), yes, if $\frac{r^2}{h^2}>\frac{2}{5}\frac{n-1}{n^2}$, where r is the radius, h the height, and $\frac{w}{w'}=n$.

8. A solid homogeneous right circular cone has its vertex fixed beneath the surface of a fluid; find the least depth of the vertex so that the equilibrium shall be stable.

Result. If $h=$ height, $r=$ radius, $w'=$ specific weight of cone, $w=$ specific weight of fluid, $x=$ depth of vertex,

$$x^4>\frac{w'h^6}{w(h^2+r^2)}.$$

9. If in the last the cone is replaced by a cylinder the submerged extremity of the axis of which is fixed, find the least depth.

Result. $$x^2 > h^2 \frac{w}{w'} - \frac{r^2}{2}.$$

(The student may verify that the results of this and the previous example can be deduced from the general expression (a) by putting $\epsilon = -\frac{\pi}{2}$ and $p = x$.)

10. If the half-cylinder of Example 11, p. 29, having assumed its position of equilibrium, is movable round an axis coinciding with the intersection of its vertical plane face with the surface of the liquid, find the condition for stability.

Result. $$\frac{r^2}{h^2} > 2\frac{w'}{w}\left(1 - \frac{w'}{w}\right).$$

57. Surfaces of Revolution. When the figure of the floating body is that of a surface of revolution, take the origin, O, of co-ordinates at its lowest point, the axis of x being vertically upwards and that of y horizontal. Then if (x, y) are the co-ordinates which determine the surface of flotation in the erect position, and (x', y') those belonging to any other parallel section, we have

$$A = \pi y^2, \quad k^2 = \tfrac{1}{4}y^2, \quad V = \pi\int_0^x y'^2\,dx';$$

hence

$$HM = \frac{y^4}{4\int_0^x y'^2\,dx'}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also, by mass-moments,

$$OH \times \pi\int_0^x y'^2\,dx' = \pi\int_0^x x'y'^2\,dx'$$

therefore

$$OM = \frac{y^4 + 4\int_0^x x'y'^2\,dx'}{4\int_0^x y'^2\,dx}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus, to determine the figure of the floating body when HM is of constant length whatever be the depth of immersion, let $HM = m$ in (1),

$$\therefore\ \tfrac{1}{4}y^4 = m\int_0^x y'^2\,dx'.$$

Differentiate both sides with respect to x; then (see Williamson's *Integral Calculus*, Chap. VI)

$$y^3\frac{dy}{dx} = my^2,$$

$$\therefore\ y^2 = 2mx,$$

which shows that the generating curve is a parabola; hence when a paraboloid of revolution floats in a liquid the height of the metacentre above the centre of buoyancy is constant for all depths of immersion.

Example.

Find the nature of the generating curve so that for the surface of revolution and for all depths of immersion the height of the metacentre above the lowest point shall be any assigned function of the height of the section of flotation.

Let $OM = \phi\,(x)$ in (2); then, writing ϕ instead of $\phi\,(x)$ for shortness,

$$\tfrac{1}{4}y^4 + \int_0^x x'y'^2\,dx' = \phi\int_0^x y'^2\,dx' \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating with respect to x, and putting p for $\dfrac{dy}{dx}$,

$$py^3 + xy^2 = y^2\phi + \frac{d\phi}{dx}\int_0^x y'^2\,dx \quad . \quad . \quad . \quad . \quad (2)$$

Dividing out and again differentiating,

$$\frac{d}{dx}\left[\frac{py^3 + (x-\phi)y^2}{\dfrac{d\phi}{dx}}\right] = y^2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is the differential equation of the required generating curve.

If, for instance, the metacentre is at a constant height, a, above the lowest point, we know that the curve is a circle, and this appears at once from (2), since $\phi = a$,

$$\therefore py + x = a, \quad \therefore ydy + (x-a)dx = 0, \quad \therefore (x-a)^2 + y^2 = a^2.$$

58. Metacentric Evolute. Suppose a body of given mass to float in a liquid; then if we consider all possible displacements round a principal axis of the area of flotation —and not merely small displacements—in which the volume of the displaced liquid is constant, the lines of action of the forces of buoyancy will envelop a certain surface fixed in the body. This surface is called the *metacentric evolute* for the given displaced volume.

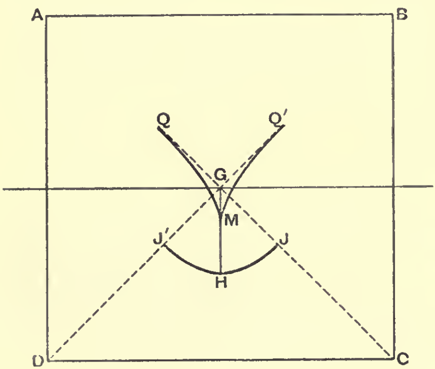

Fig. 16.

As a particular case, consider the displacements of a square board, $ABCD$, Fig. 16, floating in a liquid of double its own specific weight. The displacement is always half the volume of the board; and when the board floats with AB horizontal, the centre of buoyancy is H, the metacentre being M, such that $HM = \frac{1}{3}a$, where $2a = AB$. In this position the equilibrium is unstable. The curve of buoyancy for positions intermediate to those in which the surfaces of flotation are DB and CA is the portion $J'HJ$ of a parabola whose parameter is $\frac{2}{3}a$. The lines of action of the forces of buoyancy are always normals to this parabola, and their envelope is the evolute, QMQ', of the parabola. The positions in which DB and CA are in the surface of the fluid are positions of stable equilibrium, the metacentric heights GQ and GQ' being each $\frac{\sqrt{2}}{3}a$.

In general, for the displacements of any body in one plane—the volume of the displaced liquid being constant—the metacentric evolute is the evolute of the curve of buoyancy in the plane.

59. Stability in two Fluids. Let $DAOB$, Fig. 17, represent a body floating partly in a homogeneous fluid of specific weight w' and partly in one of specific weight w, the latter being the lower, and suppose the position of equilibrium to be found. We may evidently imagine the volume, DAB, of the upper fluid completed by adding the portion AOB, and all the forces in play will be those due to an immersion of the whole volume in a fluid of specific weight w' and an immersion of the portion AOB in one of specific weight $w - w'$.

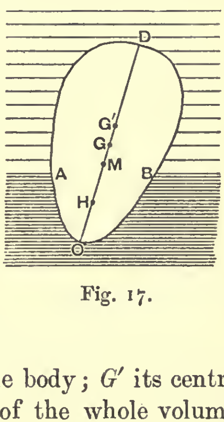

Fig. 17.

Let G be the centre of gravity of the body; G' its centre of volume, i. e. the centre of gravity of the whole volume supposed to be homogeneously filled; H the centre of volume of the portion in the lower fluid before displacement; M the metacentre corresponding to this lower fluid (of specific weight $w - w'$); V the volume of the lower and V' that of the upper fluid displaced. The position of M is given by the equation $HM = \frac{Ak^2}{V}$.

For simplicity we have assumed G, G' and H in the original position to lie on the same vertical line; but the method of investigating any case in which they are not thus simply situated will be readily understood from the simple case supposed.

The points G', G, M, H may in any individual case have relative positions different from those represented in the figure.

We suppose the displacement to be made round a principal diameter of the section AB through its centroid, in which case the wedge forces of buoyancy at the section are equivalent to a couple, whose moment in the present instance is $Ak^2(w-w')$.

The equilibrium will be stable if the sum of the moments of the forces acting on the body in its position of displacement round an axis perpendicular to the plane of displacement, drawn through G or through any other convenient point, is in a sense opposed to that of the displacement.

Now, if W = weight of body, the forces in action are W acting down through G, together with $V(w-w')$ acting up through M, and $(V+V')w'$ up through G'. The sum of their moments about H in the sense opposed to the angular displacement is

$$\theta\,\{V(w-w').HM+(V+V')\,w'.HG'-W.HG\},$$

and if the expression in brackets is positive, the equilibrium is stable.

It is sometimes more convenient to take the restoring moment about the lowest point, O, of the axis of the body.

In the above expression we may put

$$W = Vw + V'w';$$

and it is evident that if the centre of gravity, G, of the body coincides with its centre of volume, G', the condition becomes simply $HM > HG$—as is evident *a priori*.

EXAMPLES.

1. A circular cone the length of whose axis is 20 inches is formed of two substances whose specific gravities are 3 and 8, the denser forming a cone whose axis is 12 inches long and the other forming the frustum which completes the whole cone; it is immersed, vertex downwards, in a liquid whose specific gravity is 14, on top of which rests a liquid whose specific gravity is 1, the whole cone being immersed; show that for stable equilibrium the radius of the base must be less than 11·64 inches.

2. A solid homogeneous prism whose section perpendicular to its edge is an isosceles triangle of height h and base $2a$ rests, vertex downwards, in two liquids, the first of thickness c and specific weight w_1, the second heavier and of specific weight w_2, that of the prism being w; find the condition for stability.

Result. $$w_1(c+x)^3+(w_2-w_1)x^3>w\frac{h^5}{a^2+h^2},$$

x being given by the equation

$$w_1(c+x)^2+(w_2-w_1)x^2=wh^2.$$

It is assumed that part of the prism stands above the first liquid and that part projects into the second. If the value of x given by the equation is inconsistent with these assumptions the physical conditions of the problem are altered. See an analogous case at p. 77, vol. i.

60. Floating Vessel containing Liquid. Suppose a vessel, represented in Fig. 18, to contain a given volume of liquid of specific weight w' and to float in a liquid of specific weight w. If the vessel receives a small angular displacement, there will be a force of buoyancy due to the external fluid acting upwards through its metacentre M; the line of action of the weight of the contained fluid acts through

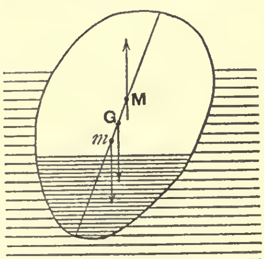

Fig. 18.

its new centre of gravity and it intersects the line GM in m, the metacentre of this contained fluid. This force acts downwards, and W, the weight of the vessel acting through its centre of gravity, G, also acts downwards. The weight of the internal fluid may assist either in promoting stability or in promoting instability according to the position of m. If, as in the figure, m is below G, this force promotes stability. If $V =$ volume of displaced external fluid, $V' =$ volume of internal fluid, the restoring moment is

$$\theta\,(wV\,.\,GM + w'V'\,.\,Gm).$$

Examples.

1. Find the least height to which a uniform heavy cylindrical vessel of negligible thickness can be filled with water so that when it is placed with its axis vertical in water the equilibrium may be stable.

Result. Let h be the distance of the centre of gravity of the vessel from the base, W the weight of the vessel, and A the area of the base; then the least height to which it can be filled is

$$h - \frac{W}{2Aw}.$$

2. If the cylinder contains a liquid of specific weight w' and floats in a liquid of specific weight w, with its axis vertical, find the condition of stability.

Result. Let $w' = n\,.\,w$, $W = Acw$, and $x =$ the height to which the cylinder is filled; then, for stability, the expression

$$2n(n-1)x^2 + 4ncx + 2c^2 - (n-1)r^2 - 4ch$$

must be positive.

3. If a uniform hollow cone of negligible thickness contains a liquid of specific weight w' and floats in a liquid of specific weight w with its axis vertical and vertex downwards, find the condition of stability.

Result. If x is the length of the axis occupied by the internal fluid, y the length occupied by the external fluid, h the whole length of the axis, l the distance of the centre of gravity of the cone from the vertex, r = radius of base, $w' = nw$, W = weight and V = volume of cone, and if $W = m \cdot Vw$, we have

$$y^3 - nx^3 = mh^3,$$

and for stability the expression

$$3\left(1 + \frac{r^2}{h^2}\right)(y^4 - nx^4) - 4mh^3 l$$

must be positive.

4. A thin vessel in the form of a surface of revolution contains a given quantity of homogeneous liquid and rests with its vertex at the highest point of a rough curved surface: find the condition of stability for small lateral displacements.

Result. Let W be the weight of the vessel (without the liquid), h the distance of its centre of gravity from the vertex, V the volume of the liquid, w its specific weight, z the distance of its centre of gravity from the vertex, A the area of the free surface of the liquid, k the radius of gyration of this area about its diameter of displacement, ρ and ρ' the radii of curvature of the vessel and the fixed surface in the plane of displacement; then the restoring moment is proportional to

$$(Vw + W)\rho - \left(1 + \frac{\rho}{\rho'}\right)(Vwz + Ak^2w + Wh),$$

and if this expression is positive, the equilibrium is stable. The restoring moment is equal to this expression multiplied by $\frac{\rho'\theta}{\rho + \rho'}$, where θ is the small angular displacement of the vessel. (See *Statics*, vol. ii, Art. 279, 4th ed.)

5. In the last example find the position of the metacentre.

Ans. If H is the centre of gravity of the contained fluid,

$$HM = \frac{Ak^2}{V} + \frac{W}{Vw}\left(h - \frac{\rho\rho'}{\rho + \rho'}\right).$$

6. If the vessel is a paraboloid of revolution resting on a horizontal plane, the weight of the liquid being P and the latus rectum of the parabola $4a$, the condition for stability is

$$W(2a - h) > \tfrac{2}{3}P\left(\frac{P}{2\pi a w}\right)^{\frac{1}{2}}.$$

7. A thin hollow conical vessel has its vertex fixed at a depth x below the surface of water, and mercury is poured into the vessel; find the volume of the mercury when the equilibrium becomes unstable.

Result. Let W be the weight of the vessel and l the distance of its centre of gravity from the vertex; $r =$ radius of base and $h =$ height of vessel, w and σ the specific weights of the water and mercury, $y =$ height of cone of mercury; then the equilibrium will be stable so long as

$$y^4 < \frac{w}{\sigma} x^4 - \frac{4 W h^4 l}{\pi r^2 (r^2 + h^2) \sigma}.$$

8. If instead of pouring a liquid into the vessel a solid cone of specific gravity σ, just fitting the hollow cone, is dropped into it, find the greatest height of this cone consistent with stability.

Result. $$y^4 < \frac{w}{\sigma}\left(1 + \frac{r^2}{h^2}\right) x^4 - \frac{4 W h^2 l}{\pi r^2 \sigma}.$$

9. A thin hollow vessel in the form of any surface of revolution has its vertex fixed at a depth x below the surface of water, and mercury is poured into the vessel; find the volume of the mercury when the equilibrium becomes unstable.

Result. Let O be the vertex of the vessel, A the area of the section of flotation, G and H the centres of gravity of the vessel and of buoyancy, A' the area of the surface of the mercury, k' its radius of gyration about its diameter, H' the centre of gravity of the mercury; then

$$w(Ak^2 + V \,.\, OH) - \sigma(A'k'^2 + V' \,.\, OH') - W \,.\, OG$$

must be positive, V and V' being the volumes of displaced water and of mercury.

When the equation of the generating curve is given, V, V', A, A', &c., will be known.

10. A cylindrical vessel of radius r is movable round a smooth horizontal axis through its centre of gravity G which is at a height h above the base. Water is gradually poured into the vessel, the height of the water being x at any instant. Prove that the height, y, of the metacentre above G is given by the equation

$$y = \frac{x}{2} + \frac{r^2}{4x} - h.$$

Taking two rectangular axes, Ox, Oy, construct the hyperbola represented by this equation, and show that there is for some time a state of instability, succeeded by one of stability, and finally a state of instability.

(Produce yO to H so that $OH = h$; then H is the centre of the hyperbola; take K on Ox so that $OK = 2h$; then the asymptotes are Hy and HK. Let the hyperbola cut Ox in A and B, OA being $< OB$; the equilibrium is unstable from $x = 0$ to $x = OA$; stable from $x = OA$ to $x = OB$; unstable afterwards.)

11. If the cylinder is replaced by a prismatic trough, show that the equilibrium is stable for values of

$$x < \tfrac{3}{2} h \cos^2 a$$

and unstable for greater, a being the semivertical angle.

61. Stability in Heterogeneous Fluid. We shall now suppose that a body floats in a fluid of variable density which is subject to the action of gravity. The level surfaces of the external force being horizontal planes, these planes will also be surfaces of constant density. Hence if w is the specific weight of the fluid at any point whose depth below the free surface, LN, Fig. 19, of the fluid is ζ, we have

$$w = f(\zeta). \quad . \quad (1)$$

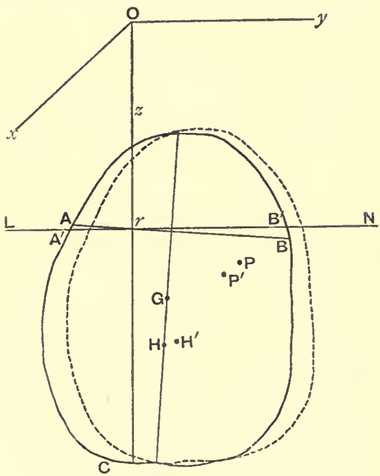

Fig. 19.

Suppose the dotted curve to represent the original position of the floating body, and that the full curve ACB represents its position when it has received a slight angular displacement, θ, round any assigned horizontal line Ox—which we suppose to be perpendicular to the plane of the paper.

Take the vertical plane through the original line joining G and H, the centres of gravity of the body and of buoyancy, which is perpendicular to Ox as plane of yz, the point, O, in which this plane cuts Ox being taken as origin, the vertical Oz as axis of z, and the horizontal line, Oy, perpendicular to Ox as axis of y. Thus the displacements of all points of the body take place in planes parallel to the plane of yz.

The section of flotation of the body in the displaced position is represented by $A'B'$. Suppose AB to be the section of the body made by the plane LN in the original position; and in this position let b be the distance between the line GH and the axis Oz. Let h be the height of O above LN.

The equation of the plane $A'B'$ is $z-h=0$, and this was the equation of AB in the original position; but by rotation in the sense indicated in the figure the equation of AB in its displaced position becomes $z-\theta y-h=0$, and therefore the old and new positions of the plane AB—and of every plane horizontal section of the body — intersect on the axis Oz.

Suppose P' to be any point in the body whose original position was P (the latter point being supposed to be marked in fixed space and not in the body); and let x, y, z be the co-ordinates of P with reference to the fixed axes at O.

Then the co-ordinates of P' are $(x, y-\theta z, z+\theta y)$, so that the density of the fluid which would exist at P' if the body were removed would be, by (1),

$$f(z+\theta y-h), \text{ or } f(\zeta+\theta y), \text{ or } w+\theta y\frac{dw}{dz}, \quad . \quad . \quad . \quad (2)$$

since the depth of P' below the surface LN is $z+\theta y-h$.

Hence when the element of volume $dxdydz$ at P is carried to P', it will experience a force of buoyancy

$$\left(w+\theta y\frac{dw}{dz}\right)dxdydz; \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and since the points P are all those included within the original volume, BCA, immersed, the corresponding forces of buoyancy will omit the wedge $B'rB$ and include the wedge ArA'—the latter not, in reality, contributing any force of buoyancy at all in the displaced position, while the former does. We must therefore specially include the wedge $B'rB$ and exclude ArA'.

Let w_0 be the specific weight of the fluid at the surface LN; let dS be the area of any element of the surface AB (such as that represented at P in Fig. 8); then if y_0 is the distance of this element from the line through r parallel to Ox, the volume of the small cylinder standing on dS, as in Fig. 8, is $\theta y_0 dS$. Also let x_0 be the x co-ordinate of the element dS, and let c be the original depth of G below the horizontal plane xOy. Then we have, in their new positions,

the co-ordinates of P ......$x, \quad y-\theta z, \quad z+\theta y,$

,, ,, ,, G ......$0, \quad b-\theta c, \quad c+\theta b,$

,, ,, ,, dS ......$x_0, \; y_0-\theta h, \; h+\theta y_0.$

Now we shall calculate the sum, L, of the moments of the forces of buoyancy round the horizontal axis through G parallel to Ox in the sense opposite to that of the displacement, i. e. counterclockwise as we view the figure. If a force having components X, Y, Z acts at the point (x, y, z), its moments round axes through the point (α, β, γ) parallel to the axes are $Z(y-\beta)-Y(z-\gamma)$, and two similar expressions (*Statics*, vol. ii, Art. 202).

In the present case only the z-component of force exists, and this at P' is the expression (3) with a negative sign, while at the new position of the surface element dS it is

$$-\theta w_0 y_0 dS. \qquad (4)$$

Hence we have

$$L = \iiint\left(w+\theta y\frac{dw}{dz}\right)\{y-b-\theta(z-c)\}\,dx\,dy\,dz$$
$$+\theta w_0\int y_0(y_0-b)\,dS. \qquad (5)$$

Now observe that we neglect θ^2; also if W is the weight of the volume ACB of fluid originally displaced,

$$\iiint wy\,dx\,dy\,dz = W.b,$$

since the y of H was originally b. Hence the term independent of θ in (5) disappears, as it must, of course; and we have

$$L = \theta\iiint(y^2-by)\frac{dw}{dz}\,dx\,dy\,dz-\theta\iiint w(z-c)\,dx\,dy\,dz$$
$$+\theta w_0\int(y_0{}^2-by_0)\,dS. \qquad (6)$$

Observe also that w is a function of z alone, so that the first triple integral can be written in the form

$$\int\left[\iint(y^2-by)\,dx\,dy\right]\frac{dw}{dz}\,dz, \qquad (7)$$

and if A denotes the area of any section for which z is constant (i.e. any section of the body parallel to AB), k the radius of gyration of this section round the line in its plane parallel to Ox, at the point where Oz cuts the section, and $\bar{y}$ the distance of the 'centre of gravity' of the area from this same line, the double integral in the brackets in (7) is

$$A(k^2-b\bar{y}), \qquad (8)$$

so that the first integral in (6) is

$$\theta\int A(k^2-b\bar{y})\frac{dw}{dz}\,dz. \qquad (9)$$

The second integral in (6) is $-\theta W.HG$, and the last is $\theta w_0 A_0(k_0^2 - b\bar{y}_0)$, where k_0 is the radius of gyration of the section, AB, of flotation (whose area is A_0) round the line through r parallel to Ox, and $\bar{y}_0$ is the distance of the 'centre of gravity' of the section from this line. Hence (6) becomes

$$\frac{L}{\theta} = \int A(k^2 - b\bar{y})\frac{dw}{dz}dz + w_0 A_0(k_0^2 - b\bar{y}_0) - W.HG. \quad . \quad (10)$$

For stability this must be a positive moment; and in the particular case in which w is constant and the displacement is made round a diameter of the section AB, it is obvious that we get the same condition as in Art. 54.

But the forces of buoyancy will also, in general, produce a moment round the horizontal axis through G parallel to Oy, i.e. a moment tending to turn the body across the plane of displacement. If M is this moment, we have

$$M = \iiint \left(w + \theta y \frac{dw}{dz}\right) x\,dx\,dy\,dz + \theta w_0 \int x_0 y_0 dS. \quad . \quad (11)$$

Let P denote the product of inertia, $\iint xy\,dx\,dy$, of any section round axes in its plane parallel to Ox and Oy at the point where the section is cut by Oz; then

$$\frac{M}{\theta} = \int P\frac{dw}{dz}dz + w_0 P_0. \quad . \quad . \quad . \quad . \quad (12)$$

This moment will not exist if P is zero for all sections, or if the fluid is homogeneous and P is zero for the surface of flotation.

Let us now calculate the *work done* in the displacement of the body round Ox.

The work which would be done on a material system by force the components of whose intensity at (x, y, z) are

X, Y, Z for any small displacement whose typical components are $\delta x, \delta y, \delta z$ is

$$\int (X\delta x + Y\delta y + Z\delta z)\, dm \,; \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and if the displacement is produced by small rotations, $\delta\theta_1$, $\delta\theta_2, \delta\theta_3$, round the axes of co-ordinates, we have $\delta x = y\delta\theta_1 - x\delta\theta_2$, with similar values of δy and δz. Hence, if L, M, N are the typical moments of the force intensity about the axes, the work is

$$L\delta\theta_1 + M\delta\theta_2 + N\delta\theta_3. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

In the present case the only rotation is that about Ox, $\therefore \delta\theta_2 = \delta\theta_3 = 0$. Consider the moment L as that of the forces of buoyancy in the displaced position ACB, and calculate the element of work done by these forces in any *further* small displacement by which the angle θ is increased by $d\theta$. Then the infinitesimal element of work done in this further displacement is

$$L\, d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

But (taking the forces of buoyancy alone),

$$L = -\iiint \left(w + \theta y \frac{dw}{dz}\right)(y - \theta z)\, dx\, dy\, dz - \theta w_0 \int y_0^2\, dS \quad . \quad (16)$$

$$= -Wb - \theta \left\{ \int Ak^2 \frac{dw}{dz}\, dz - W(c + HG) + w_0 A_0 k_0^2 \right\} \quad . \quad (17)$$

$= -Wb - K\theta$, suppose ; . . . . (18)

and the integral of this expression from $\theta = 0$ to $\theta = \theta$ expresses the work done by the forces of buoyancy in the displacement from the initial position of the body (represented by the dotted contour) to that represented by ACB. Hence the work is

$$-Wb\theta - \tfrac{1}{2} K\theta^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The work done by the weight of the body is simply $W\delta\bar{z}$, in which $\delta\bar{z}$ must be accurate as far as θ^2; i.e. $\delta\bar{z} = b\theta - \frac{1}{2}c\theta^2$. Hence the work done by all the forces is

$$-\frac{\theta^2}{2}\left\{\int Ak^2\frac{dw}{dz}dz - W.HG + w_0A_0k_0^2\right\}, \quad . \quad . \quad (20)$$

and this, with reversed sign, is the work which must be done *against* the forces to produce the displacement.

EXAMPLES.

1. If a solid homogeneous cone float, vertex down, in a fluid in which the density is directly proportional to the depth, find the condition of stability.

Result. If h' is the length of the axis immersed and h is the height of the cone, the equilibrium will be stable if $\cos^2\alpha < \frac{4h'}{5h}$, where α is the semivertical angle of the cone.

2. Determine the condition of stability of a solid homogeneous cylinder in the same circumstances.

Result. If r is the radius, h the height of the cylinder, and h' the length of the axis immersed, the condition of stability is

$$r^2 > h'(h - \tfrac{2}{3}h').$$

3. If a spherical balloon of weight B is held at a given height by a rope made fast to the ground, find the work done in displacing it about the ground end of the rope through a small angle.

Result. If h is the height of the centre of the balloon and W the weight of the displaced air, the work is

$$\tfrac{1}{2}\theta^2(W - B)h.$$

CHAPTER III

GENERAL EQUATIONS OF PRESSURE

62. Equation of Equilibrium of a Fluid under Gravity. If in the case of a fluid acted upon solely by gravity we imagine the density not to be the same at all points, the expression (α), Art. 10, for the intensity of pressure will no longer hold. For in Fig. 12, Art. 10, the weight of the cylindrical column PN will not be wzs, since w varies from point to point of its depth. But if w is the specific weight at any depth z, the weight of this cylinder is $s\int wdz$, the limits of z being 0 and NP; and, as before, this weight must be equal to the upward pressure on the base at P, viz. $p \cdot s$. Hence

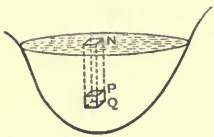

Fig. 20.

$$p = \int wdz$$

$$\therefore \frac{dp}{dz} = w. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If, for example, the density varies directly as the depth, we have $w = kz$, and (1) gives

$$p = \frac{1}{2} kz^2.$$

Equation (1) could have been obtained by considering the equilibrium of a very small rectangular parallelepiped, PQ, Fig. 20, described with vertical and horizontal sides at P. For, if s = area of each horizontal face, and if Q is a point vertically below P so that $PQ = dz$, the weight of the element is $w \cdot sdz$, where w is the weight per unit volume of the fluid at P. Also the downward pressure on the horizontal face at P is $p \cdot s$, where p is the pressure intensity at P; and since the pressure intensity at Q is $p + \frac{dp}{dz}dz$, the upward pressure on the horizontal face at Q is $\left(p + \frac{dp}{dz}dz\right)s$. Considering the separate equilibrium of this elementary parallelepiped, and resolving forces vertically, we have

$$\left(p + \frac{dp}{dz}dz\right)s = p \cdot s + w \cdot sdz,$$

$$\therefore \quad \frac{dp}{dz} = w,$$

as before.

If we are measuring force in pounds' weight and length in inches, p will be in pounds' weight per square inch, z being the depth of the point in inches, and w the weight per cubic inch of the liquid at P in pounds' weight—in other words, w is the number of pounds mass of the liquid per cubic inch at P.

If force is measured in poundals, the weight per cubic inch of the liquid at P is about $32{\cdot}2\,w$, where w is still the number of pounds mass per cubic inch at P.

It is usual to denote the number of units of mass per unit volume by ρ. If then force is measured as a multiple of the weight of the unit mass, the equation for p is

$$\frac{dp}{dz} = \rho. \qquad \qquad (2)$$

But if force is measured in absolute units, the number of these in the weight of a unit mass being g (i. e. 32·2 poundals or 981 dynes, according as the 'British Absolute' or the C. G. S. system is used), w in (1) is ρg, and the equation for p is

$$\frac{dp}{dz} = \rho g. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

With this form of the equation, and the C. G. S. system of units, the student must observe that—

p is in dynes per square centimetre,
z „ linear centimetres,
ρ „ grammes per cubic centimetre,
g „ centimetres per second per second (about 981).

If the fluid is of constant density, (1) gives $p = wz$, the result which in the previous chapters we have employed in the case of water.

In the case of a gas ρ, or w, is proportional to p: and, as in Art. 33,

$$p = 2926{\cdot}9 \frac{T}{s} \rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where p is the intensity of pressure in grammes' weight per square centimetre, T is the absolute temperature of the gas on the Centigrade scale, s is the specific gravity of the gas referred to air, and ρ is the mass of the gas in grammes per cubic centimetre.

Using equation (2), and denoting $2926{\cdot}9\frac{T}{s}$ by k, we have

$$\frac{dp}{p} = \frac{dz}{k}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

z being measured vertically downwards. If z is measured vertically upwards, we have

$$\frac{dp}{p} = - \frac{dz}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Integrate this, assuming T constant throughout the gas, and suppose that when $z = 0$ the value of p is p_0; then

$$p = p_0 e^{-\frac{z}{k}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This gives the intensity of pressure at a height of z centimetres in the atmosphere, on the assumption of constant temperature, s being unity, and p_0 being the intensity of pressure at the ground.

Suppose any gas contained in a pipe, vertical or not, closed at the upper end. Let O be the point at which $z=0$ and p is p_0; let P be any point above O; let the pipe at O be open to the air, so that p_0 is produced by the atmosphere in contact with the gas at O. At the point P in the pipe the intensity of pressure of the enclosed gas is given by (7), and at P just outside the pipe the intensity of the pressure, p_1, of the air is given by the equation

$$p_1 = p_0 e^{-\frac{z}{k_1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $k_1 = 2926{\cdot}9\ T$, the gas and the air being assumed to be at the same temperature.

Now if the gas is lighter than air—suppose hydrogen or coal gas—k is $> k_1$, and therefore

$$p > p_1,$$

i. e. the gas would escape into the surrounding air if an aperture were opened in the pipe at P. At O the gas does not rush out of the pipe, although a communication is established with the air; the gas at O would *diffuse* into the air, but we may suppose that the pipe at O contains a piston which restrains the gas and on the top of which the atmosphere presses.

The higher the point P in the pipe, the greater the ratio of p to p_1, and therefore the more rapid the escape of the gas when a communication is opened. Hence the gas lights at the top of a house are, if the taps are opened to the same extent, brighter than those at the bottom of the house; and, in consequence of this, it is commonly said that 'the pressure of the gas at the top of the house is greater than that at the bottom'—a thing which could not possibly be true, since gravitation must diminish the pressure as the height increases. It is not the pressure of the gas that is greater at the top but the velocity of its escape.

When a balloon ascends, the neck is, for safety, left open to the air, so that the intensity of pressure of the gas at the neck is that of the atmosphere at this point; the gas does not rush out at the neck; but if a valve is opened at the top of the balloon, the gas will escape for the reason already given—viz. that the intensity of pressure of the enclosed gas at this point is greater than that of the adjacent air.

If the gas in the pipes were heavier than air, p would be $< p_1$, and the reverse of the above would be true.

When density is measured in pounds per cubic foot, intensity of pressure in pounds' weight per square foot, and T is $460 + t$, the absolute temperature on the Fahrenheit scale,

$$p = 53 \cdot 3 \frac{T}{s} \rho, \quad \ldots \ldots \quad (9)$$

and at a height of z feet in the column of gas we have (7), in which k has the value given in (9).

Since T will usually be a large number, if z does not exceed one or two hundred feet, we may take $e^{-\frac{z}{k}} = 1 - \frac{z}{k}$, and we have

$$p - p_1 = p_0 \frac{z(1-s)}{53 \cdot 3 \times T}, \quad \ldots \ldots \quad (10)$$

for the excess of gas pressure in the pipes over that of the outside air at a height of z feet; and in this equation the pressures may be estimated in any units whatever.

EXAMPLES.

1. Find the pressure on a plane vertical area placed in a slightly compressible fluid.

Adopting the units of the C. G. S. system,

$$\frac{dp}{dz} = g\rho, \quad \ldots\ldots\ldots \quad (1)$$

at a point P whose depth is z cm. Also, if k is the resilience of volume, or modulus of cubical compressibility,

$$k = \rho\frac{dp}{d\rho}, \quad \therefore \frac{k}{\rho}d\rho = dp,$$

$$\therefore \frac{k}{\rho}\frac{d\rho}{dz} = g\rho,$$

$$\therefore \rho = \frac{\rho_0}{1 - \frac{g\rho_0}{k}z} = \rho_0\left(1 + \frac{g\rho_0}{k}z\right),$$

where ρ_0 is the density at the surface, and $\frac{1}{k^2}$ is neglected.

Substituting this in (1), we have

$$p = g\rho_0\left(z + \frac{g\rho_0}{2k}z^2\right). \quad \ldots\ldots\ldots \quad (3)$$

Let A be the magnitude of the given area, and $\bar{z}$ the depth of its centre of area.

If dS is the element of area at P, the whole pressure is $\int pdS$; and if $\int z^2dS$, which is the moment of inertia of the area about the line AB (Fig. 5) in which its plane intersects the surface, is denoted by $A\lambda^2$, while $\int zdS = A\bar{z}$, the resultant pressure is

$$Ag\rho_0\left(\bar{z} + \frac{g\rho_0}{2k}\lambda^2\right)$$

in dynes. Dividing this expression by g, we have the pressure in grammes' weight.

2. Find the position of the centre of pressure on any vertical area which is symmetrical with respect to its principal axes at its centre of area, when immersed in a slightly compressible fluid.

With the notation of p. 10, and denoting the radius of gyration of the area about the line AB (Fig. 5) by λ, we have

$$\xi = -\frac{k_2^{\,2}}{h}\cos\alpha\left\{1 + \frac{g\rho_0}{2kh}(2h^2-\lambda^2)\right\},$$

$$\eta = -\frac{k_1^{\,2}}{h}\sin\alpha\left\{1 + \frac{g\rho_0}{2kh}(2h^2-\lambda^2)\right\}.$$

3. Assuming the resilience of volume of sea-water to be, in C.G.S. units, $2{\cdot}33 \times 10^{10}$, and that 1 mile = 160933 centimetres, find the fractional increase in density at a depth of 1 mile in the ocean.

From the equations

$$\rho\frac{dp}{d\rho} = 2{\cdot}33 \times 10^{10}$$

$$\frac{dp}{dz} = 981\,\rho,$$

we have

$$2{\cdot}33 \times 10^{10}\frac{d\rho}{\rho^2} = 981\,dz,$$

$$\therefore\ \frac{\rho-\rho_0}{\rho_0} = \frac{981\,\rho_0 z}{2{\cdot}33 \times 10^{10} - 981\,\rho_0 z},$$

where ρ_0 is the density at the surface. Taking $\rho_0 = 1{\cdot}026$, we find at the depth of a mile

$$\frac{\rho-\rho_0}{\rho_0} = \frac{1}{142{\cdot}8},\ \text{nearly.}$$

4. Assuming the resilience of volume of sea-water to be constant at all depths, find what the depth of the ocean should be at a point where the density of the water is double the surface density.

Result. Nearly 71·92 miles.

5. Represent graphically the densities of sea-water at points on a vertical line drawn downwards from the surface.

Let O be the point on the surface, OA the vertical line drawn downwards to the point, A, at which the density would be doubled; produce OA to C so that $OA = AC$; draw a horizontal line, CH, through C. Then if the densities at various points on OC are represented by ordinates drawn at these points perpendicularly to OC, their extremities trace out a hyperbola whose centre is C and asymptotes CO and CH.

6. If the density of a fluid varies as any given function of the depth, find the depth of the centre of pressure on a plane vertical area.

Ans. If $\rho = f'(z)$, the depth of P is $\dfrac{\int z f(z)\, dS}{\int f(z)\, dS}$.

63. General Equations of Equilibrium. If the forces acting on the fluid are any assigned system, let the force per unit mass at P have for components parallel to any three rectangular axes the values X, Y, Z, so that on an element of mass dm these forces will be Xdm, &c. At P draw a small rectangular parallelepiped, with edges Pa, Pb, Pc, or dx, dy, dz, parallel to the co-ordinate axes. Then, if ρ is the density of the fluid at P, the mass contained in this parallelepiped is $\rho\, dx\, dy\, dz$.

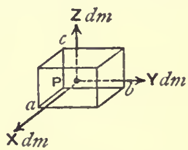

Fig. 21.

Consider the separate equilibrium of this fluid. If p is the pressure-intensity at P, the pressure on the face bPc is $p \,.\, dy\, dz$, and since the pressure-intensity on the opposite face is $p + \dfrac{dp}{dx} \,.\, dx$, the pressure on the face is

$$\left(p + \frac{dp}{dx}\, dx\right) dy\, dz.$$

For the equilibrium of the element, equate to zero the component of force acting on it parallel to the axis of x, and we have

$$Xdm + p.dydz - \left(p + \frac{dp}{dx}dx\right)dy\,dz = 0,$$

$$\text{or } \frac{dp}{dx} = \rho X. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly

$$\frac{dp}{dy} = \rho Y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{dp}{dz} = \rho Z, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

by resolving forces parallel to the other axes.

Since the direction of the axis of x is any direction, the equation $\frac{dp}{dx} = \rho X$ asserts that the differential coefficient of p in any direction is equal to the product of the density and the force-intensity in that direction; so that if S is the force-intensity along any line denoted by s at the point P,

$$\frac{dp}{ds} = \rho S. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Thus, if the position of P is expressed in polar co-ordinates,

$$\frac{dp}{dr} = \rho R, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

where R is the force per unit mass at P along the radius vector.

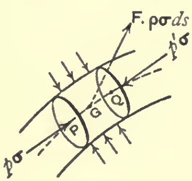

Fig. 22.

This may also be seen in the following manner.

Let PQ, Fig. 22, be an element, ds, of length of any curve through P; round this as an axis describe a cylinder of small uniform cross-section, σ; consider the separate equilibrium of the fluid contained within this cylindrical element of volume. If F is the external force per unit mass exerted on the fluid in the neighbourhood of P, the force on the enclosed fluid is $F.\rho\sigma ds$; if p is the pressure-intensity at P and p' $\left(\text{which is } p + \frac{dp}{ds}ds\right)$, the pressure-intensity at Q, the forces on the ends of the cylinder at P and Q are $p\sigma$ and $p'\sigma$. In addition to these there are side pressures which are all at right angles to the axis PQ.

Resolving forces along PQ, we have

$$p\sigma - p'\sigma + F.\rho\sigma\, ds \,.\cos\theta = 0,$$

where θ is the angle between F and PQ; and this is the same as

$$\frac{dp}{ds} = \rho F \cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We have already pointed out (Art. 4) the essential difference between the pressure-intensity, p, at a point P in a perfect fluid and the stress at a point in a strained solid—namely, that p has no reference to any *direction* at the point, but is the same in magnitude for every element-plane at the point. Hence p is a function of the co-ordinates, x, y, z, of P, i.e., $p = f(x, y, z)$, or, in other words,

dp is a perfect differential of a function of co-ordinates; and the quantities ρX, ρY, ρZ are the differential coefficients of $f(x, y, z)$; that is,

$$dp = \rho X dx + \rho Y dy + \rho Z dz\,; \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

and since $\frac{d}{dy}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(\frac{df}{dy}\right)$, X, Y, Z must satisfy the conditions

$$\frac{d}{dy}(\rho X) = \frac{d}{dx}(\rho Y)\,;\ \frac{d}{dy}(\rho Z) = \frac{d}{dz}(\rho Y)\,;\ \frac{d}{dz}(\rho X) = \frac{d}{dx}(\rho Z),$$

which give, by eliminating ρ, the condition

$$X\left(\frac{dY}{dz} - \frac{dZ}{dy}\right) + Y\left(\frac{dZ}{dx} - \frac{dX}{dz}\right) + Z\left(\frac{dX}{dy} - \frac{dY}{dx}\right) = 0, \quad . \quad (\delta)$$

and this means simply that the expression

$$X\,dx + Y\,dy + Z\,dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\epsilon)$$

either is a perfect differential of a function of x, y, z, or is capable of being made so by multiplication by a factor. In fact, since, as we have seen, dp is a perfect differential of a function of co-ordinates, the density ρ is such an integrating factor of (ϵ). Unless, then, the force components satisfy the condition (δ), no fluid can be in equilibrium under the action of forces of the type X, Y, Z.

In particular, a fluid can always be in equilibrium under the action of forces directed towards fixed centres and proportional to any powers of the distances of a point from these centres. If the fluid is a liquid of constant density throughout, we must have $\frac{dX}{dy} = \frac{dY}{dx}$, &c., that is, the expression (ϵ) is itself a perfect differential; and this means that the space occupied by the liquid can be mapped out by a series of surfaces such that at each point of any one surface the resultant force is normal to the surface; or, again, that forces have a potential—that is, that the work done by the forces in moving a particle from any point P to any other point Q is the same by whatever path the particle is allowed to travel from P to Q.

If the fluid is a gas, $p = k\rho$, and (γ) becomes

$$\frac{dp}{p} = \frac{1}{k}(X\,dx + Y\,dy + Z\,dz), \quad . \quad . \quad . \quad . \quad (\zeta)$$

where k involves the temperature of the gas at P; and if the temperature is the same throughout, (ϵ) must be a perfect differential, as for a liquid of constant density.

If the applied forces have a potential, V, the expression (ϵ) is dV, and (γ) gives

$$dp = \rho\,.\,dV, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\eta)$$

and of course the condition (δ) is satisfied, even though the density varies from point to point.

In all cases in which the external forces have a potential, their *level surfaces*, or equipotential surfaces (*Statics*, vol. ii, Art. 327), are also surfaces of equal pressure of the fluid; for (η) shows that in passing from a point P to another close point such that $dV = 0$, we have $dp = 0$. If the density of the fluid is variable, it will be constant all over a level surface of the external forces; for, since in (η) the left side is a perfect differential of a function of x, y, z, the right side must be so, and this requires that ρ is some function of V, i.e.

$$\rho = f(V), \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

so that at all points for which V is constant ρ is also constant.

For a slightly compressible fluid, whose resilience of volume is k, equation (η) becomes for forces having a potential

$$k\frac{d\rho}{\rho^2} = dV, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and for a gas

$$\frac{dp}{p} = \frac{1}{k}dV. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If the temperature of the gas varies, $p = c\rho(1 + at)$, where c is a constant, t is the temperature at any point in the gas, and a is a constant. Hence, in general,

$$\frac{dp}{p} = \frac{Xdx + Ydy + Zdz}{c(1 + at)}, \quad . \quad . \quad . \quad . \quad (8)$$

and if the applied forces have a potential, V,

$$\frac{dp}{p} = \frac{dV}{c(1 + at)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

so that, since the left-hand side is a perfect differential and the right side must also be one, it is necessary that t should be some function of V; in other words, t is constant along each equipotential surface of the external forces. Hence for a gas subject to any conservative system of forces (i. e. forces having a potential) each level surface of the forces is at once a surface of constant pressure-intensity and a surface of constant temperature.

64. Non-conservative Forces. If $Xdx + Ydy + Zdz$ is not a perfect differential (but of course satisfying the condition (δ), p. 71), and if at any point, P, in the fluid we describe the surface of constant pressure, whose equation is

$$p = \text{const.}, \qquad (1)$$

this surface will not coincide with the surface drawn through P along which the density is constant, i. e. the surface whose equation is

$$\rho = \text{const.} \qquad (2)$$

These two surfaces will intersect in some curve, which is called the curve of constant pressure and constant density at the point P.

We propose to find the direction of this curve at P.

Let l, m, n be the direction-cosines of the tangent to the curve at P; then if ds is the indefinitely small element of its length between P and a neighbouring point Q; and if $p = f(x, y, z)$ at P, the value of p at Q is

$$f(x + lds,\, y + mds,\, z + nds),$$

$$\text{i.e. } p + \left(l\frac{dp}{dx} + m\frac{dp}{dy} + n\frac{dp}{dz}\right) ds.$$

Hence, since there is no change in the value of p,

$$l\frac{dp}{dx} + m\frac{dp}{dy} + n\frac{dp}{dz} = 0, \qquad (3)$$

or, by the general equations of equilibrium,

$$lX + mY + nZ = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

so that PQ is at right angles to the direction of the resultant force at P.

Similarly, since there is no change in ρ from P to Q, we have

$$l\frac{d\rho}{dx} + m\frac{d\rho}{dy} + n\frac{d\rho}{dz} = 0; \quad . \quad . \quad . \quad . \quad (5)$$

and therefore from (4) and (5) we have

$$l : m : n = Y\frac{d\rho}{dz} - Z\frac{d\rho}{dy} : Z\frac{d\rho}{dx} - X\frac{d\rho}{dz} : X\frac{d\rho}{dy} - Y\frac{d\rho}{dx}. \quad (6)$$

Now, denoting by λ, μ, ν the components of the curl of the force (p. 92), i. e.

$$\left.\begin{aligned} \lambda &= \frac{dZ}{dy} - \frac{dY}{dz}, \\ \mu &= \frac{dX}{dz} - \frac{dZ}{dx}, \\ \nu &= \frac{dY}{dx} - \frac{dX}{dy}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

the equations of p. 71 give

$$Z\frac{d\rho}{dy} - Y\frac{d\rho}{dz} + \rho\lambda = 0, \quad . \quad . \quad . \quad . \quad (7)$$

$$X\frac{d\rho}{dz} - Z\frac{d\rho}{dx} + \rho\mu = 0, \quad . \quad . \quad . \quad . \quad (8)$$

$$Y\frac{d\rho}{dx} - X\frac{d\rho}{dy} + \rho\nu = 0. \quad . \quad . \quad . \quad . \quad (9)$$

These last show that (6) become

$$l : m : n = \lambda : \mu : \nu, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and the differential equations of the curve are

$$\frac{dx}{\lambda} = \frac{dy}{\mu} = \frac{dz}{\nu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

from which, by integration, the equations of these curves are found.

Hence the direction of any such curve at any point coincides with the direction of the curl of the external force.

If the fluid is a gas whose temperature varies from point to point, we have $p = c\rho\,(1 + a\,t)$, where t is the temperature at P, and c and a are constants. Now the previous result is absolutely general, whatever be the connexion between p and ρ; and if p and ρ are both constant along any curve, t must also be constant along the curve.

When in any case the components of the external force per unit mass are assigned—of course satisfying the necessary condition, p. 71—there will be several laws of density which permit the fluid to be in equilibrium. In fact, ρ may be any of the integrating factors of the expression $Xdx + Ydy + Zdz$. We shall illustrate this in some of the following examples.

Examples.

1. A mass of homogeneous liquid of density ρ attracting itself according to the law of nature surrounds a homogeneous sphere of density σ and radius r, also attracting the liquid; find the pressure-intensity at each point.

Let P be a point in the liquid at a distance r from the centre, O, of the sphere, and let forces be measured in absolute units. Then the resultant force per unit mass at P is

$$\tfrac{4}{3}\pi\gamma\rho\,.\,r + \tfrac{4}{3}\pi\gamma\,(\sigma - \rho)\,\frac{a^3}{r^2},$$

where γ is the constant of gravitation.

Hence
$$\frac{dp}{dr} = -\,\tfrac{4}{3}\pi\gamma\rho\left\{\rho r + (\sigma - \rho)\,\frac{a^3}{r^2}\right\},$$

$$\therefore p = -\tfrac{2}{3}\pi\gamma\rho\left\{\rho r^2 - 2\,(\sigma - \rho)\,\frac{a^3}{r}\right\} + C.$$

Let b be the external radius of the bounding surface of the liquid, and suppose that $p = 0$ at this surface; then

$$p = \tfrac{2}{3}\pi\gamma\rho\left\{\rho(b^2 - r^2) + 2(\sigma - \rho)a^3\left(\frac{1}{r} - \frac{1}{b}\right)\right\}.$$

2. If a plane at a distance c from the centre, $c > a$, is drawn in the liquid, find the whole amount of pressure on one side of this plane.

Let the foot of the perpendicular from the centre on the plane be A; describe a circle of radius x about A; then $2\pi x dx$ is a ring at each point of which p has the value above; and the whole pressure on the plane is $2\pi\int p x dx$. Now $r^2 = c^2 + x^2$, $rdr = xdx$; therefore the pressure is

$$2\pi\int_c^b prdr, \text{ or}$$

$$\tfrac{1}{3}\pi^2\gamma(b-c)^2\rho\left\{\rho(b+c)^2 + 4(\sigma - \rho)\frac{a^3}{b}\right\}.$$

3. A gas contained in a vessel is acted upon by gravity and also by a force emanating from a fixed vertical axis and proportional to the distance from that axis; find the surfaces of equal pressure.

Let force be measured in absolute units; then the action of gravity is ρg per unit volume. Let the axial force at a distance ζ from the axis be $\mu a\frac{\zeta}{c}\cdot\rho$ per unit volume. Here c is a constant length, a is an acceleration, and μ is a number.

Let z be the distance of any point P in the gas measured downwards from some fixed horizontal plane. Then the equations of pressure in the directions of ζ and z are

$$\frac{dp}{d\zeta} = \mu a\frac{\zeta}{c}\rho; \quad \frac{dp}{dz} = g\rho,$$

or, since $p = k\rho$,

$$\frac{d\log p}{d\zeta} = \frac{\mu a}{kc}\zeta; \quad \frac{d\log p}{dz} = \frac{g}{k}.$$

Integrating the first,

$$\log p = \frac{\mu a}{2kc}\zeta^2 + f(z).$$

where $f(z)$ is some function of z; and the second gives

$$\frac{df}{dz}=\frac{g}{k}, \quad \therefore \; f(z)=\frac{g}{k}z+C,$$

where C is a constant.

Hence the complete value of p is given by the equation

$$\log p = \frac{\mu a}{2kc}\zeta^2+\frac{g}{k}z+C,$$

where C depends on the position of the plane from which z is measured. Let the free surface of the gas be everywhere subject to the pressure-intensity p_0, which may be that of the atmosphere, let O be the point in which this surface is met by the axis from which ζ is measured, and let z be measured from the horizontal plane at O. Then when $\zeta = 0$ and $z = 0$, $p = p_0$, $\therefore C = \log p_0$, and

$$\log\frac{p}{p_0}=\frac{\mu a}{2kc}\zeta^2+\frac{g}{k}z.$$

Hence when p is constant we have

$$\zeta^2=-\frac{2cg}{\mu a}z+\text{const.},$$

which shows that the surfaces of constant pressure are paraboloids of revolution round the axis of ζ, the latus rectum being $\frac{2gc}{\mu a}$

The case imagined here is the same as if a gas which does not mix with the air is contained in a vessel which revolves round a vertical axis. After a while, however, the gas would no longer continue to be unmixed, since diffusion would take place. We must remember that a gas is not a body which is, in the strictest sense, at rest; its molecules are in a state of perpetual (and even violent) motion, so that diffusion into any other gas in contact with it must take place. To a much smaller extent the same is true for two liquids in contact.

4. Determine the conditions under which force having components per unit mass represented by

$$\mu(y^2+yz+z^2),\ \mu(z^2+zx+x^2),\ \mu(x^2+xy+y^2)$$

can keep a fluid at rest.

(This is a purely academic question, to which nothing real corresponds, and it is given merely as an illustration of mathematical method which might be applied in various cases of fluids of variable density.)

The equation of pressure is

$$dp = \mu\rho[(y^2+yz+z^2)dx+(z^2+zx+x^2)dy+(x^2+y^2+z^2)dz] \quad (1)$$

The expression in brackets satisfies the condition (δ), p. 71; that is, it is capable of being made a perfect differential by being multiplied by a factor, or various factors; and these factors will represent laws of density for which equilibrium is possible.

The process of treatment is as follows. Let us seek the integral of the equation

$$(y^2+yz+z^2)dx+(z^2+zx+x^2)dy+(x^2+xy+y^2)dz = 0, \quad . \quad (2)$$

that is, some function of x, y, z which when differentiated will give, not precisely the left-hand side of (2), but this expression multiplied by some factor. First suppose z constant, and integrate (2), which becomes

$$\frac{dx}{\left(x+\frac{z}{2}\right)^2+\frac{3}{4}z^2} + \frac{dy}{\left(y+\frac{z}{2}\right)^2+\frac{3}{4}z^2} = 0;$$

$$\therefore \tan^{-1}\frac{2x+z}{z\sqrt{3}} + \tan^{-1}\frac{2y+z}{z\sqrt{3}} = C',$$

$$\text{or} \quad \frac{x+y+z}{z^2-2xy-zx-zy} = C, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where C is a constant. Now replace C by $\phi(z)$, where ϕ is some unknown function, and determine ϕ so as to satisfy (2).

Differentiating the equation

$$\frac{x+y+z}{z^2-2xy-2zx-2zy} = \phi(z), \quad . \quad . \quad . \quad . \quad (4)$$

and, for simplicity, denoting the denominator of the left side by D,

$$\frac{2}{D^2}\left\{(y^2+yz+z^2)dx+(z^2+zx+x^2)dy + \frac{x^2+y^2-z^2-2xz-2yz}{2}dz\right\} = \phi'(z)dz. \quad . \quad . \quad (5)$$

With the aid of (2) this becomes

$$-\frac{(x+y+z)^2}{D^2}dz = \phi'(z)\,dz;$$

and by (4)

$$-\phi^2(z) = \phi'(z), \text{ or } \frac{d\phi}{\phi^2} + dz = 0,$$

$$\therefore \phi(z) = \frac{1}{z+k},$$

where k is an arbitrary constant. Hence the integral (4) is

$$\frac{x+y+z}{z^2 - 2xy - zx - zy} = \frac{1}{z+k},$$

$$\therefore \quad 2(xy+yz+zx) = k(x+y+z),$$

$$\text{or} \quad \frac{xy+yz+zx}{x+y+z} = \frac{k}{2}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now differentiating the left-hand side of (6) we have

$$d\,\frac{xy+yz+zx}{x+y+z} \equiv \frac{1}{(x+y+z)^2} \cdot \{(y^2+yz+z^2)\,dx + (z^2+zx+x^2)\,dy + (x^2+xy+y^2)\,dz\}.$$

We see then that (1) becomes

$$dp = \mu\rho \,.\, (x+y+z)^2 \,.\, d\,\frac{xy+yz+zx}{x+y+z};$$

and if the right-hand side is a perfect differential, we must have

$$\rho = \frac{C}{(x+y+z)^2} \cdot f\left(\frac{xy+yz+zx}{x+y+z}\right),$$

where f is any function and C is a constant. This expresses all possible laws of density. In particular, we can have

$$\rho = \frac{C}{(x+y+z)^2},$$

i.e., the density at any point is inversely as the square of the perpendicular from the point on the plane $x+y+z=0$. Also we can have $f(u) \equiv \frac{1}{u^2}$, so that $\rho = \frac{C}{(xy+yz+zx)^2}$ gives also a possible law of density.

The surfaces of constant pressure are hyperboloids given by the equation
$$xy+yz+zx = C(x+y+z),$$
and those points at which both density and pressure are constant are the curves of intersection of these hyperboloids and planes
$$x+y+z=k.$$

5. Find laws of density for which the equilibrium of a fluid under the action of force components proportional to
$$(y+a)^2, \; cz, \; -c(y+a)$$
will be possible.

Result. The equation of pressure is
$$dp = \mu\rho\,(y+a)^2 \,.\, d\left(x-\frac{cz}{y+a}\right).$$

Hence, generally,
$$\rho = \frac{C}{(y+a)^2}\cdot f\left(x-\frac{cz}{y+a}\right),$$
and the surfaces of constant pressure are hyperbolic paraboloids; the curves of constant density and pressure, if we take $\rho = \dfrac{C}{(y+a)^2}$, are right lines. The value $\rho = \dfrac{C}{\{x(y+a)-cz\}^2}$ is also possible, &c.

6. Find the same things for force components
$$cy-bz, \; az-cx, \; bx-ay.$$

Result. The surfaces of constant pressure are planes passing through the line $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$. Curves of constant p and ρ are lines parallel to this.

7. Find the same things for components
$$x-a, \; -\sqrt{h^2-z^2-(x-a)^2}, \; z.$$

Result. $dp = \rho\sqrt{h^2-z^2-(x-a)^2}\,.\,d\left(\sqrt{h^2-z^2-(x-a)^2}+y\right).$

8. In a spherical mass of liquid of constant density attracting itself according to the law of nature find the pressure-intensity at any point.

Result. In Ex. 1, p. 76, let $a = 0$, or let $\sigma = \rho$, and we have
$$p = \tfrac{2}{3}\pi\gamma\rho^2(b^2-r^2).$$

9. A homogeneous liquid is acted upon by gravity and also by a force emanating from a fixed vertical axis and varying as the distance from that axis; find the surfaces of constant pressure.

Result. Paraboloids of revolution.

10. A spherical mass of liquid attracting itself according to the law of nature being supposed to arrange itself in spherical layers of constant density in such a way that the change of pressure from layer to layer is proportional to the change in the square of the density, it is required to find the density at any distance from the centre.

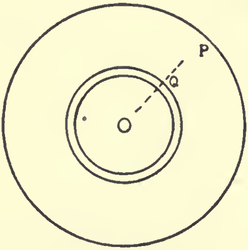

Fig. 23.

Let P, Fig. 23, be any point distant x from the centre, O; let a thin spherical layer with radius OQ, or r, be taken; and let ρ be the density of this layer. If R is the resultant force per unit mass at P, the equation of pressure is

$$dp = -\rho R dx.$$

Now we are given that $dp = \lambda d(\rho^2) = 2\lambda\rho d\rho$, where λ is constant. The attraction of the shell Q at P is equal to

$$\gamma \frac{4\pi\rho r^2 dr}{x^2},$$

so that

$$R = \frac{4\pi\gamma}{x^2}\int_0^x \rho r^2 dr,$$

since the shells outside P contribute nothing to the attraction. Hence at P we have

$$\frac{1}{\rho}\frac{dp}{dx} = -\frac{4\pi\gamma}{x^2}\int_0^x \rho r^2 dr,$$

or

$$2\lambda \frac{d\rho}{dx} = -\frac{4\pi\gamma}{x^2}\int_0^x \rho r^2 dr.$$

Denoting $\frac{2\pi\gamma}{\lambda}$ by k^2, and differentiating this with regard to x,

$$\frac{d}{dx}\left(x^2 \frac{d\rho}{dx}\right) = -k^2\rho x^2,$$

where ρ is the density at P. This equation gives

$$\frac{d^2\rho}{dx^2} + \frac{2}{x}\frac{d\rho}{dx} + k^2\rho = 0, \quad . \quad . \quad . \quad . \quad . \quad (a)$$

which is a form of Bessel's equation, and the value of ρ is $\frac{1}{\sqrt{kx}} J_{\frac{1}{2}}\, kx)$. Without resorting to the theory of Bessel functions, we see, however, that (a) can be satisfied by assuming $\rho = \frac{\sin \mu x}{x}$. Substituting this in (a), we have $\mu = k$,

$$\therefore \rho = \frac{\sin kx}{x}, \text{ or } = A\frac{\sin kx}{x},$$

where A is constant. If the density at the centre is ρ_0,

$$\rho = \rho_0 \frac{\sin kx}{kx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

The fact that k is the reciprocal of a length may be verified from the nature of γ and of λ.

The assumption that the change in pressure is proportional to the change in the square of the density is made by Laplace in discussing the figure of the Earth.

It is interesting to see how the constants ρ_0 and k in the expression (β) can be calculated from two observed facts with regard to the Earth—viz., that the density at the surface is about 2·75 (that of water being 1), and that the mean density of the globe is 5·5.

The whole mass of the globe is $4\pi\int_0^a \rho x^2 dx$, where a is the radius of the surface; and this becomes from (β)

$$\frac{4\pi\rho_0}{k^3}(\sin ka - ka \cos ka)\,; \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

and if μ is the mean density, this mass $= \frac{4}{3}\pi a^3 \mu$;

$$\therefore \mu = \frac{3\rho_0}{k^3 a^3}(\sin ka - ka \cos ka) \quad . \quad . \quad . \quad . \quad (\delta)$$

Denote ka by θ; then we have the data

$$2{\cdot}75 = \rho_0 \frac{\sin\theta}{\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\epsilon)$$

$$5{\cdot}5 = 3\rho_0 \frac{\sin\theta - \theta\cos\theta}{\theta^3}, \quad . \quad . \quad . \quad . \quad . \quad (\zeta)$$

therefore by eliminating ρ_0 we have, to find θ, the equation

$$\frac{3-2\theta^2}{3\theta} = \cot\theta, \quad \ldots \ldots \quad (\eta)$$

and an indication of the value of θ is obtained by taking axes of x and y, and laying off values of θ along Ox. We thus construct two curves,

$$y = \frac{3-2x^2}{3x} \text{ and } y = \cot x,$$

whose intersection gives the value sought.

When the curves are drawn we see that they intersect at a point between $x = 2{\cdot}4$ and $x = 2{\cdot}5$; and after a few trials we arrive at the result

$$x = 2{\cdot}4605,$$

that is,

$$k = \frac{2{\cdot}4605}{a};$$

and substituting $\theta = 2{\cdot}4605$ in (ϵ), we have

$$\rho_0 = 10{\cdot}746,$$

so that Laplace's law of density becomes

$$\rho = 10{\cdot}746 \frac{\sin\left(2{\cdot}4605\frac{x}{a}\right)}{2{\cdot}4605\frac{x}{a}}.$$

This law is found to agree very well with observations relative to the figure of the Earth. (See Thomson and Tait's *Nat. Phil.*, Part II, pp. 403, &c.) The density at the centre is therefore a little less than double the mean density.

The intensity of pressure at any point can now be calculated. We have $p = \lambda\rho^2 + C$, where C is a constant. If p_1 and ρ_1 are the pressure and density at the surface,

$$p - p_1 = \lambda(\rho^2 - \rho_1^2).$$

Now $\lambda = \frac{2\pi\gamma}{k^2}$, $k = \frac{2{\cdot}4605}{a}$, and a may be taken as 637×10^6 centimetres. The mean density of the Earth being taken as $5{\cdot}5$, we find $\gamma = \frac{1}{1496 \times 10^4}$, which is the number of dynes with which two gram spheres whose centres are 1 cm. apart attract each other. Also p in the above equations is measured in dynes per square cm. At the centre let p be p_0, and $\rho = 10{\cdot}746$. Using these numbers, we have the value of $p_0 - p_1$, and since the pressure-intensity of 1 atmosphere is $1{\cdot}014 \times 10^6$ dynes per sq. cm., the value of $p_0 - p_1$ measured in atmospheres is, approximately, 2,951,400 atmospheres, which, since p_1 is only 1, may be taken as the value of the pressure-intensity at the centre of a fluid sphere having the dimensions, mean density, and surface pressure of the Earth. This shows how enormous is the pressure-intensity—nearly 3 millions of atmospheres—at the centre of the Earth.

11. Inside a homogeneous sphere of density σ there is a spherical cavity containing liquid of density ρ, which does not fill the cavity completely; find the form of the free surface of the liquid under the attraction of the sphere.

Result. A plane perpendicular to the line joining the centres of the sphere and cavity.

[The liquid is in equilibrium under two forces, $\frac{4}{3}\pi\gamma\sigma r$ and $\frac{4}{3}\pi\gamma\sigma r'$, the first directed towards the centre of the sphere and the second directed away from the centre of the cavity; hence

$$\frac{1}{\rho}dp = \frac{4}{3}\pi\gamma\sigma(-rdr + r'dr').]$$

12. A mass of homogeneous liquid surrounds a sphere and is acted upon by the attraction of the sphere, for the inverse square law, and also by a force directed from a fixed diameter of the sphere and proportional to the distance from this axis; find the surfaces of constant pressure.

The equation of pressure is

$$\frac{1}{\rho}dp = -\frac{\mu}{r^2}dr + \mu'\zeta d\zeta$$

$$\therefore \frac{p}{\rho} = \frac{\mu}{r} + \tfrac{1}{2}\mu'\zeta^2 + \text{const.}$$

The surfaces of constant pressure are generated by the revolution round the given axis of curves of the form

$$\frac{r^3}{a^3}\sin^2\theta - \frac{r}{c} = k,$$

where a, c, k are constants.

13. A mass of fluid is acted upon by forces directed to fixed centres and varying inversely as the squares of distances from these centres; find the surfaces of constant pressure.

They are given by the equation

$$\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} + \frac{\mu_3}{r_3} + \dots = C.$$

14. Find the form of the free surface of a liquid contained within a spherical cavity inside a homogeneous sphere if, in addition to the attraction of the sphere, the liquid is acted upon by a force emanating from the line of centres and proportional to the distance from this line.

A paraboloid of revolution.

[Equation of pressure:

$$\frac{dp}{\rho} = \tfrac{4}{3}\pi\gamma\rho\,(-rdr + r'dr' + c\zeta d\zeta)].$$

15. At any point P inside a mass of liquid which is in equilibrium under its own attraction, according to the inverse square, is described a sphere of any radius r; prove that the mean value of the pressure in this sphere is less than the pressure at its centre by $\frac{2}{5}\pi\rho^2\gamma^2$.

16. A mass, m, of homogeneous liquid is in equilibrium under the action of its own attraction, according to the law of inverse square; prove that the force with which one hemisphere attracts the other is equal to that with which two spheres each of mass m would attract each other if their centres were placed at a distance $\frac{4a}{\sqrt{3}}$ apart, where a is the radius of the mass m.

17. For a fluid in motion prove the following construction for the direction of the surface of constant pressure at any point P: let f be in magnitude and direction the acceleration of a particle at P due to the external forces acting on the particle; let a be in magnitude and direction the actual acceleration of the particle; reverse a in sense, and find the resultant of the vectors f, $-a$; then this resultant is the normal to the surface of constant pressure at P. [See equation (4), p. 71.]

18. A liquid is contained in a perfectly smooth hollow sphere which is caused to rotate with angular velocity ω about a fixed

smooth horizontal axis O; prove that at any instant the surfaces of constant pressure are a system of parallel planes.

The result follows from the geometrical construction in the last example, or thus: let C be the centre of the sphere; then, all the external forces (gravity and pressures of the sphere) passing through C, the particles have no rotation relatively to C; that is, if P is any point of the liquid, the line CP remains always in the same direction. Hence if $OC = a$, and θ is at any instant the angle made by OC with the vertical, and we take the axis of revolution as that of z, and those of x and y horizontal and vertical, the co-ordinates of P at any instant are given by the equations

$$x = a\sin\theta + m,\ \ y = a\cos\theta + n,\ \ z = k,$$

where m, n, k are constants. Hence if a_x, a_y, a_z are the accelerations of P parallel to the axes, and $\frac{d\theta}{dt} = \omega$,

$$a_x = -a\omega^2\sin\theta,\ \ a_y = -a\omega^2\cos\theta,\ \ a_z = 0;$$

therefore

$$\frac{dp}{dx} = w\frac{a\omega^2}{g}\sin\theta;\ \frac{dp}{dy} = w\frac{a\omega^2}{g}\cos\theta;\ \frac{dp}{dz} = w;$$

$$\therefore\ \frac{p}{w} = \frac{a\omega^2}{g}\left[x\sin\theta + y\cos\theta + \frac{g}{a\omega^2}z\right] + \text{const.}$$

19. A spherical shell is filled with liquid and caused to rotate about a vertical diameter whose highest point is A; prove that the total vertical component of pressure on a spherical cap whose angular radius measured from A is θ is

$$\pi a^3 w(1-\cos\theta)^2\left\{\frac{1+2\cos\theta}{3} + \frac{\omega^2 a}{4g}(1+\cos\theta)^2\right\},$$

where a is the radius of the sphere.

Determine the cap such that this vertical component vanishes.

Result. If $\frac{\omega^2 a}{g} = k$, $\cos\theta = -\frac{\sqrt{4+3k}}{\sqrt{4+3k}+2}$.

If $\omega = 0$, or the liquid is at rest, the angular radius of the cap is $120°$; and when ω is not zero, the angular radius is $> 120°$. If ω increases towards ∞, θ tends to π.

65. Green's Equation. Let ABC, Fig. 24, be any closed surface; let U and V be any functions of x, y, z, the co-ordinates of any point P at which an element of volume $d\Omega$ is taken; and let ∇^2 stand for the operation

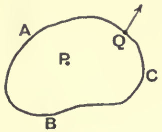

Fig. 24.

$$\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2};$$

then if we take the integral

$$\int U\nabla^2 V d\Omega$$

throughout the volume enclosed by ABC, the result can be expressed in terms of another volume-integral taken through the same space and of a surface-integral taken over the bounding surface ABC. Thus, let Q be any point on the surface at which an element of area dS is taken, and let dn be an element of the normal at Q drawn outwards into the surrounding space (in the sense of the arrow). Then we have (see *Statics*, vol. ii, chap. xvii, Section iv)

$$\int U\nabla^2 V.d\Omega=\int U\frac{dV}{dn}.dS$$
$$-\int\left(\frac{dU}{dx}\frac{dV}{dx}+\frac{dU}{dy}\frac{dV}{dy}+\frac{dU}{dz}\frac{dV}{dz}\right)d\Omega.. \quad (1)$$

In exactly the same way, if ϕ is any other function of x, y, z, we have

$$\int U\left(\frac{d}{dx}\cdot\phi\frac{dV}{dx}+\frac{d}{dy}\cdot\phi\frac{dV}{dy}+\frac{d}{dz}\cdot\phi\frac{dV}{dz}\right).d\Omega$$
$$=\int U\phi\frac{dV}{dn}\cdot dS-\int\phi\left(\frac{dU}{dx}\frac{dV}{dx}+\frac{dU}{dy}\frac{dV}{dy}+\frac{dU}{dz}\frac{dV}{dz}\right).d\Omega. \quad (2)$$

The first of these is known as Green's equation; the second is a modification made by Lord Kelvin.

By assigning to U various values (such as a constant value, the value V, &c.) we obtain (as shown in *Statics* above) various remarkable theorems with physical applications.

A most remarkable consequence of (2) is this. If ϕ and V are any two functions satisfying the equation

$$\frac{d}{dx}\cdot\phi\frac{dV}{dx}+\frac{d}{dy}\cdot\phi\frac{dV}{dy}+\frac{d}{dz}\cdot\phi\frac{dV}{dz}=0 \quad . \quad . \quad (3)$$

at all points within a closed surface, ABC, and if the value of V is assigned at every point, Q, on the surface itself, its value at each internal point, P, is determinate.

For, if possible, let there be two different functions, viz.,

$$V\equiv f(x, y, z),$$

$$V'\equiv f'(x, y, z);$$

each satisfying (3) and such that $V = V'$ at each point, Q, on the surface, while V is, of course, not equal to V' at each internal point P. Denote $V-V'$ by ξ; then ξ satisfies (3). Now employ (2) for the volume and surface of ABC, and, moreover, choose for U the value ξ. Then

$$\int\xi\left(\frac{d}{dx}\cdot\phi\frac{d\xi}{dx}+\frac{d}{dy}\cdot\phi\frac{d\xi}{dy}+\frac{d}{dz}\cdot\phi\frac{d\xi}{dz}\right)\cdot d\Omega$$

$$=\int\xi\phi\frac{d\xi}{dn}\cdot dS-\int\phi\left[\left(\frac{d\xi}{dx}\right)^2+\left(\frac{d\xi}{dy}\right)^2+\left(\frac{d\xi}{dz}\right)^2\right]\cdot d\Omega. \quad (4)$$

But each term under the integral on the left-hand side vanishes, and the surface-value of ξ which enters into each term of the first integral on the right also vanishes; therefore the second integral on the right vanishes; but since each term of this integral is a square, we must have each term equal to zero, i.e.,

$$\frac{d\xi}{dx}=0, \quad \frac{d\xi}{dy}=0, \quad \frac{d\xi}{dz}=0,$$

must hold for all points inside ABC; and this requires that ξ is constant for all internal points, and therefore zero, since it is zero at the surface points.

Hence there cannot be two functions, V, V', satisfying (3), agreeing at each surface point, while differing at internal points. If, therefore, any *one* function, $f(x, y, z)$, of the co-ordinates is known to satisfy (3) and to have at each point on the surface an assigned particular value, it is the only one applicable to the points enclosed by the surface.

The application of this result to the case of fluid pressure is obvious. If at each point of any fluid-mass the external forces satisfy the equation

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=0, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

equations (1), (2), (3), p. 70, give

$$\frac{d}{dx}\cdot\frac{1}{\rho}\frac{dp}{dx}+\frac{d}{dy}\cdot\frac{1}{\rho}\frac{dp}{dy}+\frac{d}{dz}\cdot\frac{1}{\rho}\frac{dp}{dz}=0. \quad . \quad . \quad (6)$$

Hence if ABC is the surface of a foreign body immersed in the fluid, the distribution of the fluid which could, under the influence of the given external forces, statically replace the body is determinate since the value of the pressure-intensity is assigned at each surface point, Q. At each internal point, P, the pressure-intensity is determinate, and if ρ is, for the given fluid, a given function of p—say $f(p)$—the value of p at P is given by the equation

$$\frac{dp}{f(p)}=dV, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where V is the potential function of the external forces, involving the co-ordinates of P. This is the result referred to in p. 69, vol. i.

EXAMPLE.

If in the midst of a mass of fluid which is not self-attracting there is a solid body which attracts its own particles and those of the fluid according to the law of inverse square of distance, and if the surface, A, of this body is one of constant potential, prove that the intensity of pressure p, of the fluid at any point, P, is less than the intensity of pressure, p_0, at any point on A by an amount given by the equation

$$p = p_0 - \frac{1}{4\pi\gamma M} \cdot \int \rho R^2 d\Omega,$$

where γ is the constant of gravitation (*Statics*, vol. ii, Art. 315), M is the mass of the solid body, ρ is the density of the fluid at any point at which the attraction per unit mass due to the body is R, $d\Omega$ is an element of volume, and the integration extends over the space included between the surface A and the equipotential surface, S, described through P.

In Green's equation (1) for U choose $p - p_0$ and let V be the potential at any point due to the solid body. Then we have

$$\int (p-p_0)\nabla^2 V . d\Omega = \int (p-p_0) \frac{dV}{dn} dS - \int \left(\frac{dp}{dx}\frac{dV}{dx} + \ldots\right) d\Omega, \quad (1)$$

in which the surface-integral on the right is taken over the surface A and over the surface S, and the element of normal dn is drawn into the space *outside* the volume enclosed by A and S; this space is therefore the *interior* of the solid body and the exterior of S, so that dn in the integration over A is measured towards the interior of M.

Now we know that (*Statics*, vol. ii, Art. 329) $\nabla^2 V = 4\pi\gamma\rho'$, where ρ' is the density of the *attracting matter* (to which V is due) at the point to which V applies; and as there is none of this attracting matter at any of the points within the volume (that included between A and S) included in the integration, $\nabla^2 V = 0$. Again, at every point on the surface of A we have $p - p_0 = 0$, therefore the part of the surface-integral on the right which relates to the surface A is zero. Further, at every point on S p is constant; hence the surface-integral is simply

$$(p-p_0)\int \frac{dV}{dn} dS,$$

and is confined to the surface S. Moreover, at every point in the fluid $\frac{dp}{dx} = \rho X$, &c., and $X = \frac{dV}{dx}$; hence (1) becomes

$$0 = (p - p_0)\int \frac{dV}{dn} dS - \int \rho\, R^2 d\Omega. \quad . \quad . \quad . \quad (2)$$

Now (*Statics, ibid.*)

$$\int \frac{dV}{dn} dS = -4\pi\gamma M, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

so that the required result follows at once from (2) and (3).

66. Line-Integrals and Surface-Integrals. If any directed magnitude, or vector, has for components u, v, w along three fixed rectangular axes, the magnitude which has for components λ, μ, ν along these axes where

$$\frac{dw}{dy} - \frac{dv}{dz} = \lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{du}{dz} - \frac{dw}{dx} = \mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{dv}{dx} - \frac{du}{dy} = \nu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

has been called the 'curl' of the given vector by Clerk Maxwell. (In the theory of Stress and Strain, and in the motion of a fluid, it is convenient to define the curl as having the *halves* of the above components.)

Any vector and its curl possess the following fundamental relation: *the line-integral of the tangential component of any vector along any closed curve is equal to the surface-integral of the normal component of its curl taken over any curved surface having the given curve for a bounding edge.* (See *Statics*, vol. ii, Art. 316, *a*.)

If l, m, n are the direction-cosines of the normal at any point of such a surface, and dS the area of a superficial element at that point, while ds denotes an element of length of the bounding edge of the surface, the theorem is expressed by the equation

$$\int(l\lambda + m\mu + n\nu)\,dS = \int\left(u\frac{dx}{ds} + v\frac{dy}{ds} + w\frac{dz}{ds}\right)ds. \quad . \quad (4)$$

Now we are often given the components of curl, λ, μ, ν, and from these we require to determine the vector from which they arise. In view of such a problem, the following fact is useful. If we can find, by any means, some particular values, u_0, v_0, w_0, of the components of the required vector which will satisfy the equations (1), (2), (3), the *general* values of u, v, w are simply

$$u = u_0 + \frac{d\phi}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$v = v_0 + \frac{d\phi}{dy}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$w = w_0 + \frac{d\phi}{dz}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where ϕ is any function whatever of x, y, z. This is evident, because if we substitute u_0, v_0, w_0 for u, v, w in (1), (2), (3), we have, by subtraction,

$$\frac{d(w - w_0)}{dy} = \frac{d(v - v_0)}{dz}$$

and two analogous equations; and these signify that the expression

$$(u - u_0)\,dx + (v - v_0)\,dy + (w - w_0)\,dz$$

is a perfect differential of some function of x, y, z. If this function is denoted by ϕ, we have the results (5), (6), (7).

Of course it is a necessity from (1), (2), (3) that any possible components of curl of a vector should satisfy the identity

$$\frac{d\lambda}{dx} + \frac{d\mu}{dy} + \frac{d\nu}{dz} \equiv 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus, it is not possible to determine a vector the components of whose curl are x, y, z; but it is possible to determine one whose components of curl are x, y, $-2z$. The values $u_0 = \frac{3}{2}yz$, $v_0 = -\frac{1}{2}zx$, $w_0 = \frac{1}{2}xy$ will give these latter components of curl; so will the values $u_1 = yz$, $v_1 = -zx$, $w_1 = 0$. But it is obvious that if ϕ denotes the function $\frac{1}{2}xyz$, these two sets of components are related thus

$$u_0 = u_1 + \frac{d\phi}{dx}, \quad v_0 = v_1 + \frac{d\phi}{dy}, \quad \&c.$$

In the line-integral along the closed curve the vector whose components are $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$, $\frac{d\phi}{dz}$ may be rejected, if ϕ is a single-valued function.

Stokes's method of determining values of u, v, w from the given values of λ, μ, ν will be found in Lamb's *Hydrodynamics*, 3rd ed., p. 199.

Examples.

1. Given any unclosed curved surface in a heavy homogeneous liquid, is it possible to express the total component of pressure, on one side of the surface, parallel to—

(*a*) a horizontal line,
(*b*) a vertical line,

by an integral taken along the bounding edge of the surface?

Ans. The first is possible, but not the second. If a horizontal line is drawn at the surface of the liquid, which is taken as the plane of x, y, the component in the first case is $\int lz\,dS$, and this $= -\frac{1}{2}\int z^2 \frac{dy}{ds} ds$. This result is evident from elementary principles; because, if through the edge of the surface we describe a horizontal cylinder whose generators are parallel to the axis of x, and take a section of this cylinder perpendicular to its axis, the horizontal component of the pressure on the curved surface is equal to the pressure on this plane section, and is therefore independent of the shape and size of the given surface.

2. Given any unclosed curved surface in a heavy homogeneous liquid, is it possible to express the sum of the moments of the pressures, on one side of the surface, about—

(*a*) a horizontal line,
(*b*) a vertical line,

by an integral taken along the bounding edge of the surface?

Ans. The second is possible, but not the first; for,

$$\int (mzx - lyz)\, dS = \tfrac{1}{2}\int z^2 \left(x\frac{dx}{ds} + y\frac{dy}{ds}\right) ds;$$

and the result also follows from elementary principles, by closing the surface with a fixed cap described on the bounding edge, and then imagining the given surface to vary in size and shape, while retaining its bounding edge.

67. Equations of Equilibrium in Polar and Cylindrical Co-ordinates. Let P, Fig. 25, be any point in a fluid, at which the components of force acting on the fluid, per unit mass, are X, Y, Z parallel to the rectangular axes, Ox, Oy, Oz; and let the position of P be defined by the usual polar co-ordinates, viz. the radius vector $OP\ (= r)$, the colatitude, $POz\ (= \theta)$, and the longitude, $xOn\ (= \phi)$, this last being the angle between the plane xz and the plane containing P and the axis of z. The arcs in the figure are those determined on a sphere whose centre is O and radius OP by the axes and the line OP.

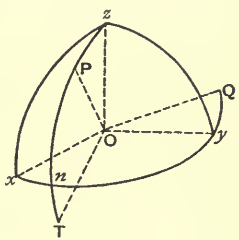

Fig. 25.

Sometimes it is convenient to consider the resultant force per unit mass at P as resolved into three rectangular components corresponding to radius vector, latitude, and longitude, i. e. components along OP, along the line at P perpendicular to OP in the plane POz, and along the tangent at P to the parallel of latitude.

Producing the great circle zPn to T so that $nT = zP = \theta$, the second of these directions is parallel to OT; and since the third is at right angles to the plane POz at P, if we produce the arc xy to Q so that $yQ = xn = \phi$, the line OQ is parallel to the tangent at P to the parallel of latitude.

Let R be the component of the force-intensity (i. e. force per unit mass) at P in the direction OP; let Θ be its component in the second and Φ its component in the third of these directions.

Now, the axis of x being in any direction, we have proved the equation

$$\frac{dp}{dx} = \rho X,$$

so that if ds is the element of length of a curve drawn in any direction at P, and S the force-intensity along the tangent to this curve at P in the sense in which ds is measured, it follows that

$$\frac{dp}{ds} = \rho S.$$

Taking ds along OP, we have $ds = dr$; taking ds along the meridian zP at P, we have $ds = rd\theta$; and taking ds along the parallel of latitude at P in the sense OQ, we have $ds = r \sin\theta \, d\phi$, since the radius of the parallel of latitude is $r \sin\theta$. Hence the equations of equilibrium are

$$\left.\begin{aligned} \frac{dp}{dr} &= \rho \, . \, R, \\ \frac{dp}{d\theta} &= \rho r \, . \, \Theta, \\ \frac{dp}{d\phi} &= \rho r \sin\theta \, . \, \Phi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Equations of equilibrium in Cylindrical Co-ordinates. By the cylindrical co-ordinates of P are meant the distance, z, of P from the plane xy, the perpendicular distance, ζ, of P from the axis of z, and the longitude, ϕ, i.e. the angle between the plane xz and the meridian plane zOP. Hence if Z, Z_1, Φ denote the components of force-intensity at P parallel to Oz, perpendicular to Oz, and along the tangent to the parallel of latitude,

$$\left.\begin{aligned} \frac{dp}{dz} &= \rho Z, \\ \frac{dp}{d\zeta} &= \rho Z_1, \\ \frac{dp}{d\phi} &= \rho\zeta\Phi. \end{aligned}\right\} \qquad (2)$$

68. Homogeneous Revolving Spheroid. The components of attraction of a homogeneous ellipsoid of revolution, of mass M,

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}=1 \qquad (1)$$

exerted per unit (gramme) mass at a point (x, y, z) on its surface, and measured in dynes, are (see *Statics*, vol. ii, pp. 324–8)

$$X = -\frac{3\gamma M}{2c^3e^3}\left(\tan^{-1}e - \frac{e}{1+e^2}\right)x, \qquad (2)$$

$$Y = -\frac{3\gamma M}{2c^3e^3}\left(\tan^{-1}e - \frac{e}{1+e^2}\right)y, \qquad (3)$$

$$Z = -\frac{3\gamma M}{c^3e^3}\left(e - \tan^{-1}e\right)z, \qquad (4)$$

where $e^2 = \dfrac{a^2-c^2}{c^2}$ and γ is the constant of gravitation, or force in dynes exerted between two spherical gramme masses with a distance of 1 cm. between their centres. This constant γ is approximately

$$\frac{1}{1496 \times 10^4}, \quad \quad (5)$$

assuming the Earth's mean density to be 5·5.

Denoting the terms in brackets in (2) and (4) by A and C, we have

$$X = -\frac{3\gamma M}{2c^3e^3}Ax, \quad Y = -\frac{3\gamma M}{2c^3e^3}Ay, \quad Z = -\frac{3\gamma M}{c^3e^3}Cz.$$

Now assume that this ellipsoid is fluid, or was at any time fluid, and revolving with angular velocity ω round its least axis, c. Take a particle at P on the surface at a distance r from the axis of revolution. This particle has an acceleration $\omega^2 r$ directed towards the axis, and the force per unit mass at P directed towards the axis is

$$-X\frac{x}{r} - Y\frac{y}{r} + \frac{1}{\rho}\frac{dp}{dr}.$$

Hence if the particle at P has unit mass,

$$\omega^2 r = -X\frac{x}{r} - Y\frac{y}{r} + \frac{1}{\rho}\frac{dp}{dr}, \quad \quad (6)$$

$$\therefore \quad \frac{1}{\rho}\frac{dp}{dr} = -\frac{3\gamma M}{2c^3e^3} \cdot Ar + \omega^2 r \quad \quad (7)$$

Similarly, since there is no acceleration in the direction of z,

$$\frac{1}{\rho}\frac{dp}{dz} = -\frac{3\gamma M}{c^3e^3} \cdot Cz. \quad \quad (8)$$

Now p is a function of r and z only, therefore

$$dp = \frac{dp}{dr}dr + \frac{dp}{dz}dz;$$

$$\therefore \quad \frac{p}{\rho} = \left(-\frac{3\gamma MA}{2c^3e^3} + \omega^2\right)\frac{r^2}{2} - \frac{3\gamma MC}{c^3e^3}\,\frac{z^2}{2} + \text{const.} \quad (9)$$

On the surface of the ellipsoid $p = 0$ or a constant, while (1) gives

$$\frac{r^2}{a^2} + \frac{z^2}{c^2} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

while (9) gives for all points on the surface

$$\left(\frac{3\gamma MA}{2c^3e^3} - \omega^2\right) r^2 + \frac{3\gamma MC}{c^3e^3} z^2 = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence equating the ratio of the co-efficients of r^2 and z^2 in (11) to the same ratio in (10) and putting $M = \frac{4}{3}\pi\rho a^2 c$, we have

$$\frac{(3 + e^2)\tan^{-1}e - 3e}{e^3} = \frac{\omega^2}{2\pi\gamma\rho} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Denote the right-hand side of this equation by k; then for a given value of ω—that is for a given value of the time of rotation, $\frac{2\pi}{\omega}$, of the spheroid—there are two values of e, i. e., two different spheroids, or none, as can be seen by solving (12) graphically.

The equation which we have to solve (replacing e by x) is this:

$$\left(1 + \frac{3}{x^2}\right)\tan^{-1}x - \frac{3}{x} = kx \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Taking two rectangular axes[1] Ox, Oy (Fig. 26), construct the curve, $OQPR$, whose equation is

$$y = \left(1 + \frac{3}{x^2}\right)\tan^{-1}x - \frac{3}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Owing to the exceeding smallness of the values of y for small values of x (such as ·05, ·1, ·2, ...) it is desirable to represent unity along Oy by a much greater length than that which represents unity along Ox. In Fig. 26 the unit representative along Oy is four times that along Ox.

[1] This geometrical representation was brought before the Oxford Mathematical and Physical Society on December 2, 1911.

The value of y when $x=\infty$ is $\frac{\pi}{2}$, so that the curve approaches asymptotically a line parallel to Ox which is above y in the figure at a distance from Ox a little more than $\frac{3}{2}$ Oy.

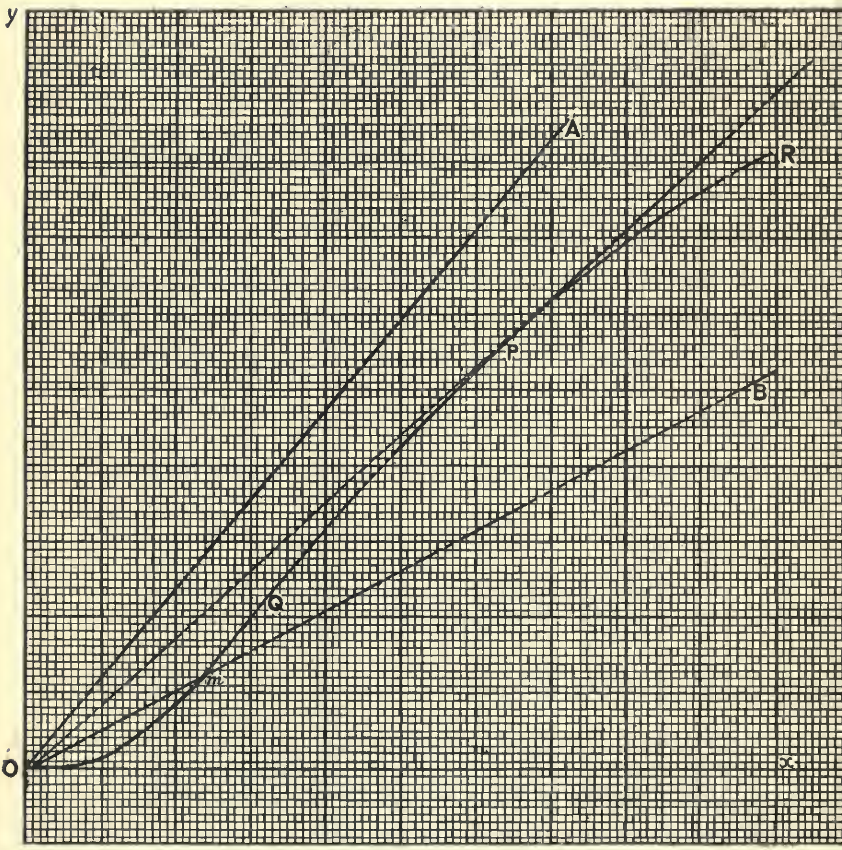

Fig. 26.

Now the values of x in (13) are given by the intersection of the curve (14) with the right line $y=kx$ (15). Hence the figure shows that there are either two solutions or none, according to the value of k. For, if from O we draw a tangent OP to the curve $OQPR$ (touching at P, the abscissa of which is very nearly 2·5293), and if the line $y = kx$ lies above OP, as OA does, there is no real point of intersection. If the line is OB, there are two points of intersection, viz., m and a distant point in which OB produced meets the branch PR produced. For the value of k which belongs to OB there is thus a small value of x and also a very large one; that is, there are two figures of equilibrium, one approaching the spherical form, and the other extremely flat. If the line does not cut the curve, the ellipsoidal form is not possible, but the body assumes, of course, some other figure.

Forming the equation of the tangent from O to the curve $OQPR$, we find that the co-ordinates (x', y') of the point of contact satisfy the equation

$$y' = \frac{4x'^3}{(x'^2+1)(x'^2+9)},$$

or that x' is given by the equation

$$\tan^{-1}x' = \frac{x'(9+7x'^2)}{(x'^2+1)(x'^2+9)},$$

which gives $x' = 2·5293$, nearly.

Thus we get the value of $\frac{dy}{dx}$ at P equal to ·22467, and this is the greatest value of k for which equilibrium is possible with an ellipsoid of revolution. Let us calculate the time of revolution, or the value of ω, which corresponds to this, assuming the density of the body to be equal to the Earth's mean density, 5·5. We have

$$\frac{\omega^2}{2\pi\gamma\rho} = ·22467; \quad \therefore \text{ if } \quad T = \frac{2\pi}{\omega},$$

$$T = 8722 \text{ seconds},$$

which is a little more than one-tenth of the Earth's present day. The value of k which corresponds to the Earth's present rotation is obtained by putting $\omega = \frac{2\pi}{86400}$; and this gives $k = \cdot 002289$.

The line OB (Fig. 26) which corresponds to this value of k is very close to Ox, and it will cut the curve at a point having a very small x—and of course another very distant point having a very large x. For the small value we can put $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5}$ in (13), and thus get $k = \frac{4}{15} x^2$.

Now $\frac{a^2}{c^2} = 1 + e^2 = 1 + x^2 = 1 + \frac{15}{4} k$; therefore

$$\frac{a}{c} = 1 + \frac{15}{8} \times \cdot 002289 = 1 \cdot 004292.$$

But the ratio of the axes for the Earth is found to be 1·003333, so that the interior of the Earth cannot be regarded as a homogeneous fluid impressing the ellipsoidal form on the thin solid crust.

When the line OB coincides with Ox, one value of e is zero and the other is ∞. The corresponding ω is zero for each case, and the two figures are a quiescent sphere and a quiescent infinitely extended thin plate.

69. Jacobi's Ellipsoids. An ellipsoid with three unequal axes is a possible figure of equilibrium of a revolving self-attracting liquid. For a homogeneous ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

if we put

$$\frac{a^2 - c^2}{c^2} = e^2, \quad \frac{b^2 - c^2}{c^2} = e'^2,$$

the components of attraction on a particle of unit mass at the surface are given by the expressions

$$X = -\frac{3\gamma Mx}{c^3}\int_0^1 \frac{u^2 du}{(1+e^2u^2)^{\frac{3}{2}}(1+e'^2u^2)^{\frac{1}{2}}} = -Ax, \text{ suppose,}$$

$$Y = -\frac{3\gamma My}{c^3}\int_0^1 \frac{u^2 du}{(1+e^2u^2)^{\frac{1}{2}}(1+e'^2u^2)^{\frac{3}{2}}} = -By,$$

$$Z = -\frac{3\gamma Mz}{c^3}\int_0^1 \frac{u^2 du}{(1+e^2u^2)^{\frac{1}{2}}(1+e'^2u^2)^{\frac{1}{2}}} = -Cz.$$

The revolving unit particle has component accelerations $\omega^2 x$ and $\omega^2 y$ in the inward directions of the axes of x and y, and the forces in these directions are $\frac{1}{\rho}\frac{dp}{dx} + Ax$ and $\frac{1}{\rho}\frac{dp}{dy} + By$; hence the equations of pressure

$$\frac{1}{\rho}\frac{dp}{dx} + Ax = \omega^2 x,$$

$$\frac{1}{\rho}\frac{dp}{dz} + By = \omega^2 y,$$

$$\frac{1}{\rho}\frac{dp}{dz} + Cz = 0;$$

$$\therefore \ \frac{p}{\rho} = \tfrac{1}{2}(\omega^2 - A)\,x^2 + \tfrac{1}{2}(\omega^2 - B)\,y^2 - \tfrac{1}{2}Cz^2.$$

If the surface of the ellipsoid is one of constant pressure, its equation must be the same as that obtained by equating p to a constant; hence

$$\frac{A-\omega^2}{C} = \frac{c^2}{a^2}, \quad \frac{B-\omega^2}{C} = \frac{c^2}{b^2};$$

therefore eliminating ω^2,

$$B - A = \frac{e^2 - e'^2}{(1+e^2)(1+e'^2)} \cdot C,$$

or, rejecting the factor $e^2 - e'^2$, the vanishing of which belongs to the previous case,

$$\int_0^1 \frac{u^2(1-u^2)(1-e^2e'^2u^2)}{(1+e^2u^2)^{\frac{3}{2}}(1+e'^2u^2)^{\frac{3}{2}}}\,du = 0. \quad . \quad . \quad . \quad (a)$$

This equation gives the relation between the two (so-called) excentricities e, e', of the principal meridians of the sought ellipsoid. All ellipsoids satisfying this equation are called Jacobi's ellipsoids, because Jacobi first pointed out the fact that a rotating liquid could assume such forms.

When e is assigned, e' must be found by trial from this equation; and the process may be assisted by a graphic method. Thus, using x instead of u, and taking rectangular axes Ox, Oy, construct the curve whose equation is

$$Y = \frac{x^2(1-x^2)}{(1+e^2x^2)^{\frac{3}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

This can easily be done by assigning for x values

$$0, \cdot 1, \cdot 2, \cdot 3, \ldots \cdot 9, 1.$$

Now if we denote the other factor under the integral sign by Z, i.e., if

$$Z = \frac{1 - e^2e'^2x^2}{(1+e'^2x^2)^{\frac{3}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

we must have

$$\int_0^1 YZdx = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\delta)$$

so that as the curve (β) is, within the limits 0 and 1 of x, entirely above the axis of x, the curve whose ordinate is YZ must be partly above and partly below the axis, i.e., Z must be negative after a certain value of x is reached.

For example, if $e = \cdot 56$ and we trace the curve (β), the shape suggests that Z should become negative somewhere in the neighbourhood of $x = \cdot 5$; and this would give $1 - \frac{1}{2}ee' = 0$, $\therefore$ $e' = 3 \cdot 57$. By tracing the curve (δ) on a conveniently large scale, we find with a planimeter that for $e' = 3 \cdot 57$ the positive part of the area is slightly in excess; while if we try $e' = 4$, the negative part prevails. Trial gives $e' = 3 \cdot 72$, nearly.

Since the value of Z must change sign for some value of x between 0 and 1, the expression $1-ee'x$ must vanish between these limits; hence ee' must be >1; and if $e=0$, e' must be ∞.

If the whole volume of the liquid is equal to that of a sphere of radius a, we must have $abc = \mathrm{a}^3$, so that

$$a = \mathrm{a}\frac{(1+e^2)^{\frac{1}{3}}}{(1+e'^2)^{\frac{1}{6}}};\ b = \mathrm{a}\frac{(1+e'^2)^{\frac{1}{3}}}{(1+e^2)^{\frac{1}{6}}};$$

$$c = \frac{\mathrm{a}}{(1+e^2)^{\frac{1}{6}}(1+e'^2)^{\frac{1}{6}}}. \quad . \quad . \quad . \quad . \quad . \quad (\epsilon)$$

Now when $e=0$, $e'=\infty$, so that $a=c=0$ and $b=\infty$; thus the ellipsoid becomes an infinitely thin circular wire of infinite length; and the corresponding value of ω is zero.

The following corresponding values of e and e' and the values of x at which the curve (γ) crosses the axis of x are calculated from results given by Sir George Darwin for Jacobi's ellipsoids in a paper published in the Proceedings of the Royal Society (Nov. 25, 1886):

e	e'	x
0	∞	0
0·204	11·390	·430
0·399	5·676	·442
0·580	3·732	·462
0·748	2·677	·499
0·932	2·147	·500
1·132	1·730	·511
1·266	1·541	·513
1·363	1·428	·514
1·393	1·393	·515

The values of e and e' become equal at 1·393, the ellipsoid being then one of revolution, at which point the ellipsoids might pursue either Maclaurin figures (of revolution) or Jacobi figures. In the latter case the values of e and e' would simply be interchanged; that is, the series of ellipsoids would be the same with the axes a and b interchanging values.

It is not evident that when $e' = \infty$ the crossing point of the curve (γ) comes in to the origin; but it can be proved that if in (a) we write p for $e^2e'^2$ and m for e'^2, the equation

$$\int_0^1 \frac{x^2\,(1-x^2)\,(1-px^2)}{(1+mx^2)^{\frac{3}{2}}}\,dx = 0,$$

when m is infinitely large, requires p to be an infinitely great number of the order $\log m$; so that, though $p = \infty$ (and therefore the x of the crossing point is zero), the ratio $p:m$ (which $= e^2$) is zero. If we retain the product $e^2e'^2$ in the denominator of (a), the result will be the same—viz., that, when m is infinitely large, no finite value of p can satisfy the equation

$$\int_0^1 \frac{x^2\,(1-x^2)\,(1-px^2)}{(1+mx^2+px^4)^{\frac{3}{2}}}\,dx = 0.$$

In order to show, however, that an ellipsoid is a possible figure of the fluid, we must prove that the corresponding value of ω^2 is positive; and this is the case when revolution is assumed to take place round the least of the principal axes, but not when it is assumed to take place round any of the others. Thus, when revolution takes place round the axis c, we have

$$\omega^2 = A - C\,\frac{c^2}{a^2} = \frac{3\,e^2\gamma M}{c^3\,(1+e^2)}\int_0^1 \frac{u^2\,(1-u^2)}{(1+e^2u^2)^{\frac{3}{2}}\,(1+e'^2u^2)^{\frac{1}{2}}}\,du,$$

which is essentially positive.

But if we assume rotation about the axis b, the components of acceleration of a particle will be $\omega^2 x$, 0, $\omega^2 z$, and the dynamical equations will be

$$\frac{1}{\rho}\frac{dp}{dx} + Ax = \omega^2 x,\ \frac{1}{\rho}\frac{dp}{dy} + By = 0,\ \frac{1}{\rho}\frac{dp}{dz} + Cz = \omega^2 z,$$

which lead to

$$\omega^2 = A - B\frac{b^2}{a^2} = C - B\frac{b^2}{c^2}.$$

Now

$$C - B\frac{b^2}{c^2} = -\frac{3e'^2\gamma M}{c^3}\int_0^1 \frac{u^2(1-u^2)}{(1+e^2u^2)^{\frac{1}{2}}(1+e'^2u^2)^{\frac{3}{2}}}\,du,$$

which is negative, $\therefore$ ω is impossible.

In the same way, rotation about the axis a is impossible.

Examples.

1. Show from equation (a) that a circular cylinder is a possible form of equilibrium.

2. Show that an elliptic cylinder rotating about its axis is a possible figure of equilibrium.

(See *Statics*, vol. ii, Art. 330. At a surface point

$$X = -4\pi\gamma\rho\frac{b}{a+b}x,\ Y = -4\pi\gamma\rho\frac{a}{a+b}y.$$

The proof then proceeds as above.)

3. Prove that an ellipsoidal figure of a rotating self-attracting liquid is not possible unless the axis of rotation coincides with one of the principal axes.

(If the direction-cosines of the axis are l, m, n, and ζ is the perpendicular from a point x, y, z on the axis, the direction-cosines of this perpendicular being λ, μ, ν, we have

$$\lambda\zeta = x - l(lx+my+nz),\ \mu\zeta = y - m(lx+my+nz),$$
$$\nu\zeta = z - n(lx+my+nz).$$

If ω is the angular velocity, the acceleration parallel to x is $\omega^2 . \lambda\zeta$;

$$\therefore \frac{1}{\rho}\frac{dp}{dx} = Ax + \omega^2 \{(m^2+n^2)x - lmy - lnz\},$$

$$\frac{1}{\rho}\frac{dp}{dy} = By + \omega^2\{-lmx + (n^2+l^2)y - mnz\}; \quad \frac{1}{\rho}\frac{dp}{dz} = \&c.$$

Then $\frac{1}{\rho} dp = \frac{1}{2}\{A + (m^2+n^2)\omega^2\}x^2 + \ldots - 2\omega^2 lmxy - \&c.,$

and as the equation of the free surface is obtained by putting $dp = 0$, and the free surface is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$, there must be no terms xy, yz, zx; that is, two of the quantities l, m, n must vanish.)

CHAPTER IV

MOLECULAR FORCES AND CAPILLARITY

70. Molecular Forces. Common observations on the resistance which solid bodies oppose to any effort to elongate or twist them have compelled physicists to assume the existence of forces between the molecules of such bodies other than the ordinary action of Newtonian gravitation.

Thus, let us fix our attention on any one molecule, m, inside a body. It is surrounded by a group of molecules, and if we take all those molecules which lie within a sphere of extremely small radius whose centre is m, there is a special action exerted on m by each molecule within this sphere, those molecules nearest to m exerting a more powerful action than those near the surface of the sphere. This holds, whatever be the sizes, the shapes, or the distances between the molecules.

Beyond a certain distance, ϵ, from m these special actions are assumed to be insensible; this length ϵ is the radius of the aforesaid sphere, called the *sphere of molecular activity*.

Now if dm and dm' are two elements of mass, *the linear dimensions of each being infinitely smaller than the length of any line from the surface of the one to that of the other*, it is assumed that these elements exert on each other a force whose magnitude is

$$f(r) \, . \, dm \, dm', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where r is the distance between the elements—i.e., the length of a line drawn from any point on one to any point on the other—and this force acts in the line joining them.

If the elements dm and dm' were homogeneous spheres, such a law of force as (1) could be assumed to hold, though their dimensions were even large compared with the distance between their centres, which distance would be the value of r in (1); but if they are not spherical, such a law could not be admitted (because it would be utterly devoid of meaning) if the elements are so close together that their linear dimensions are of the same order of magnitude as lines drawn from points on the surface of one to points on the surface of the other.

Now there are several suppositions that may be made with regard to the arrangement of matter in a body, such as the following:—

1. The matter is absolutely continuous within the volume of the body, there being no vacant spaces, however small.

2. The matter consists of molecules (in the chemical sense) which are packed very closely together, their linear dimensions being great compared with the distances between their surfaces.

3. The matter consists of molecules (in the chemical sense) which are very distant from each other, so that the space surrounding any molecule is comparatively void of matter.

If the third supposition is made, it is clear that the application of mathematical calculation becomes exceedingly difficult, if not impossible. It is true that Lamé in his *Élasticité des Corps Solides* objects strongly to the method applied by Navier and others in the theory of Elasticity, because, in applying the Integral Calculus to the determination of the action produced on a molecule of a body by the neighbouring molecules, they thus assume the continuity of matter, an assumption which Lamé describes as a 'hypothése absurde et complétement inadmissible'. His own method is a molecular one in which the existence of vacant spaces between the molecules is admitted; and the process of *integration* round a molecule is replaced by a process of mere algebraic summation—which, no doubt, is a much safer process, and should be adopted if it could be legitimately applied. It is not, however, satisfactorily applied by Lamé, since he has no hesitation in assuming a molecule to be wherever he wants one, and this assumption is not essentially different from integration.

If the second of the above suppositions is adopted, the matter surrounding a molecule, although not continuously filling space with mathematical strictness, may be assumed to be practically continuous, and the method of integration round a point becomes permissible as a very close approximation to the truth. The shapes of the molecules may possibly be such as to allow of their filling space much more effectively than if they were spheres.

But in adopting this supposition when calculating the forces produced on any molecule, m, by those within the range, ϵ, of molecular force, it will be necessary to imagine m and any *very* close neighbour, m', as *both* divided into infinitely smaller elements, of which dm is the type for the first and dm' that for the second, each of these elements being now infinitely smaller than the distance between them, and then assuming the force between them to be given by the expression (1). Thus for a pair of molecules so close that it is logically impossible to define anything that could be called the 'distance between them' we must imagine a special process of integration performed before we proceed to calculate the action of the more distant molecules within the sphere of molecular activity.

Such a process it is, of course, quite impossible to follow in detail because the form of $f(r)$ and the shapes of molecules are unknown; nevertheless, on account of the symmetry of arrangement of molecules round all points in a homogeneous body, it is possible to represent the result of such a process by a mathematical expression and to base further calculation thereon.

Various forms for $f(r)$ have been suggested, such as $A - \frac{B}{r}$ and e^{-ar}: these are, of course, merely conjectural; but it is conceivable that the observation of certain phenomena measurable *in the total* might afford a clue to, if not a necessary demonstration of, the law of this assumed molecular force.

If, then, we admit the second supposition, with the above notions, the first of our three suppositions becomes unnecessary, and Lamé's objection to the integration method loses its force.

In the study of the forms assumed by the surfaces of liquids in contact with each other and with solid bodies, it is with these molecular forces that we have chiefly to deal. Indeed, the curious forms of such surfaces become explicable on no other hypothesis than that of the existence of very intense molecular forces having an extremely small range of action.

Supposing that the force between two elements of matter is given by the expression (1), its component along any fixed line (axis of x) is

$$\frac{x' - x}{r} f(r)\, dm\, dm',$$

if the co-ordinates of dm and dm' are x and x', so that the total component force acting on dm has for expression

$$dm \int_0^\epsilon \frac{x' - x}{r} f(r)\, dm',$$

if the integration is performed with reference to r, the limits of r being 0 and ϵ. Now, since the forces are zero beyond the distance ϵ, no error is introduced by assuming r to extend to ∞, so that such an expression is often written in the form

$$dm\int_0^\infty \frac{x'-x}{r} f(r)\, dm'.$$

Some notion of the magnitude of ϵ may be obtained from experiments such as the following. Quincke covered surfaces of glass with extremely thin layers of different bodies, and on these layers then deposited drops of mercury and other liquids. Now it will be seen presently that there is a definite angle between the tangent plane to the free surface of a liquid and the tangent plane to a solid with which it is in contact; this angle is constant all round the curve in which the two surfaces intersect; and it matters not whether the solid is a millimetre or 100 millimetres thick, the value of the angle does not alter. But if the solid is, say, the millionth of a millimetre thick, the angle alters. Covering the surface of glass with a layer of sulphide of silver, Quincke found that there was no change in the angle between the surface of the drop of mercury and the plate until the thickness of the silver layer was reduced to $\frac{46}{10^6}$ mm.; and when the glass was coated with a layer of iodide of silver, no change was observed until the thickness of the layer was reduced to $\frac{59}{10^6}$ mm., or, say, one-tenth of the wave length of yellow light. These thicknesses, then, indicate the order of magnitude of the distances at which molecular attractions are sensible.

Granting the existence of these molecular forces, it follows very obviously that within a layer of a fluid just at the surface, and of the extremely small thickness ϵ, there is a special intensity of pressure which increases in magnitude as we travel from any point P (Fig. 27) along the normal

Pb to the surface, *AB*, of the fluid, towards the interior of the fluid.

For, consider a molecule of the fluid at *P*; round *P* as centre describe the sphere of molecular activity; of this only a hemisphere, *abc*, exists within the fluid, so that the molecular forces acting on the particle *P* come from the molecules of this hemisphere. Now it is obvious that the symmetrical grouping of these molecules about the line *Pb* results in their producing a resultant force on *P* inwards along *Pb*.

Describe a surface, *A′B′*, parallel to *AB* at the distance ϵ, or, *Pb*, from *AB*.

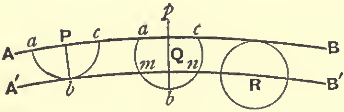

Fig. 27.

Consider now the molecular actions on a molecule *Q* anywhere within this layer. Describe round *Q* the sphere, *ambnc*, of molecular activity. Of this sphere the portion *apc* does not contain any molecules of the fluid *L*, so that the action at *Q* is due to the portion *ambnc*, and the resultant force will obviously be directed along the normal *Qb* and will be less than the force at *P*, since there is some component of force in the sense *Qp*. Finally, consider a molecule at any point *R* on the inner surface *A′B′*, and we see that since this molecule is completely surrounded by attracting molecules, there is no resultant force whatever.

Now if *F* is the force exerted at *Q* per unit mass, and *dn* denotes an element of length of the normal *Qb* at *Q* measured towards *b*, while ϖ denotes the pressure intensity at *Q* due to the forces under consideration, we have

$$\frac{d\varpi}{dn} = wF, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

w being the density of the fluid.

Since as we travel along the normal Pb from P towards b, or from Q towards b, the force F constantly preserves the sense Pb, although with diminishing value, we see that $\frac{d\varpi}{dn}$ is constantly positive, that is, ϖ continuously increases inwards until the surface $A'B'$ is reached, when F vanishes and $\frac{d\varpi}{dn} = 0$, i.e., ϖ becomes constant when we pass inwards through $A'B'$.

Hence the intensity of pressure due to molecular forces is constant throughout the interior of the fluid below $A'B'$, but it varies within the layer between AB and $A'B'$.

It is a matter of doubt with physicists whether we are or are not entitled to assume in the case of a liquid that the density within the layer contained between the surfaces AB and $A'B'$ is constant and equal to the density within the main body of the liquid. M. Mathieu, following Poisson, denies this constancy (*Théorie de la Capillarité*), but arrives, by the method of Virtual Work, at results of the same form as those obtained on the supposition of constant density.

Whatever may be the nature of the molecular forces, at any point close to the surface of separation of a liquid from another medium, we can represent the magnitude of the resultant molecular force of the liquid on a molecule m by the expression

$$m \, . \, F(\omega),$$

where ω is the area of that part of the surface of the sphere of molecular activity which exists round m within the liquid; and this force vanishes when $\omega = 4\pi\epsilon^2$. Or we might represent this resultant force (along the normal to the surface) by

$$m \, . \, F(z),$$

where z is the distance of the molecule m from the surface;

and the force $= 0$ when $z = \epsilon$. Of course the form of the function F is unknown, but it is the same at all points which are at the same normal distance from the surface, and this fact is sufficient for the purpose of calculation.

71. Calculation of Molecular Pressure. Let AB (Fig. 28) be the bounding surface of a liquid, P any point on the surface, TM the tangent plane at P, $A'B'$ the surface parallel to AB within the liquid at the depth ϵ; take an infinitely small element, σ, of area at P, and on the contour of this area describe a right cylinder, PR, extending indefinitely into the liquid. Consider now the action of molecular forces only on the liquid contained within this cylinder PR.

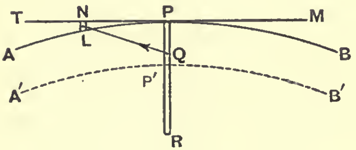

Fig. 28.

If ϖ is the intensity of molecular pressure at R, and the cylinder is terminated at R by a normal section, $\varpi\sigma$ is the pressure exerted on this section; then, for the equilibrium of the fluid in the cylinder we see that $\varpi\sigma$ must be equal to the integral of the molecular attraction exerted by the whole mass of the fluid on the portion of fluid contained in this slender canal. Now below the point P' there is no change in the molecular pressure, and there is no molecular force exerted on the elements of liquid; hence we might have taken the slender canal as reaching only to P'.

Let Q be any point in the canal between P and P' and let dm be an element of mass at Q. We shall calculate the attraction produced by the whole body of liquid below AB in the direction QR, which we know to be the direction of the resultant molecular attraction.

Now the total action on the canal PP' (or PR) can be calculated on the supposition that the liquid extends up to the tangent plane TM, and then deducting the attraction, in the sense QR, which is due to the meniscus, $ABMT'$, of liquid thus added. The attraction of this added meniscus is obviously in the sense QP, so that this must be added to the attraction of the liquid terminated by TM. Let the attraction of this fictitiously completed liquid on the canal PP' be denoted by $K\sigma$ where K is obviously the same at all points on the bounding surface AB.

Let the plane of the figure be a normal section of the surface AB making the angle θ with the principal section at P whose radius of curvature is R_1, and let the radius of curvature of the other principal section at P be R_2.

Let any point, N, be taken on the tangent TM; let $NP = x$, and on the element of area $x\,dx\,d\theta$ at N construct the small cylinder NL terminated by AB.

If w is the mass per unit volume of the liquid, the mass of this cylinder is $NL \cdot wx\,dx\,d\theta$; and if its distance, NQ, from Q is r, the molecular force which it produces on dm at Q is

$$w\,dm\,x\,dx\,d\theta\,f(r) \cdot NL. \qquad (1)$$

Now if ρ is the radius of curvature of the section AB, we have

$$NL = \frac{x^2}{2\rho}, \text{ nearly};$$

and if $PQ = z$, the component of the force (1) along QP is obtained by multiplying it by $\frac{z}{r}$; hence this component is

$$w\,dm\,\frac{zx^3\,dx\,d\theta}{2\,r\rho}\,f(r). \qquad (2)$$

Integrate this with respect to θ, keeping x and r constant.

Now (Salmon's *Geometry of Three Dimensions*, chap. xi) we know that

$$\frac{1}{\rho} = \frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2},$$

and the integration in θ from 0 to 2π makes (2) become

$$\frac{\pi}{2} w\,dm \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{zx^3\,dx}{r} f(r) \quad . \quad . \quad . \quad . \quad (3)$$

But since $r^2 = x^2 + z^2$, $x\,dx = r\,dr$, and (3) becomes

$$\frac{\pi}{2} w \left(\frac{1}{R_1} + \frac{1}{R_2}\right) dm\, z\,(r^2 - z^2) f(r)\, dr. \quad . \quad . \quad . \quad (4)$$

Let

$$f(r)\, dr = -d\phi(r), \quad . \quad . \quad . \quad . \quad . \quad (5)$$

then since $f(r)$ rapidly diminishes with an increase of r, $\phi(r)$ is a positive quantity.

The resultant action of the meniscus on dm is obtained by integrating (4) from $r = z$ to $r = \epsilon$, or to $r = \infty$. Hence the resultant is

$$\frac{\pi}{2} w \left(\frac{1}{R_1} + \frac{1}{R_2}\right) z\,dm \int_z^\infty (-r^2 + z^2)\, d\phi(r). \quad . \quad . \quad (6)$$

Now $\phi(\infty) = 0$, and

$$\int -r^2\, d\phi(r) = -r^2\phi(r) + 2\int r\,\phi(r)\, dr,$$

$$\therefore \int_z^\infty - r^2\, d\phi(r) = \quad z^2\phi(z) + 2\int_z^\infty r\,\phi(r)\, dr,$$

therefore (6) becomes

$$\pi w \left(\frac{1}{R_1} + \frac{1}{R_2}\right) z\,dm \int_z^\infty r\,\phi(r)\, dr. \quad . \quad . \quad . \quad (7)$$

Again, let

$$r\,\phi(r)\, dr = -d\psi(r), \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $\psi(r)$ is obviously positive; therefore, since $\psi(\infty)$ is evidently zero, (7) becomes

$$\pi w \left(\frac{1}{R_1} + \frac{1}{R_2}\right) z\,\psi(z)\,.\,dm. \quad . \quad . \quad . \quad . \quad (9)$$

Now put dm at Q equal to $w\sigma dz$, and (9) becomes

$$\pi w^2 \sigma \left(\frac{1}{R_1} + \frac{1}{R_2}\right) . z\psi(z)\, dz, \quad . \quad . \quad . \quad . \quad (10)$$

and the integral of this from $z = 0$ to $z = PP'$, or $= \infty$, is the total action of the meniscus on the canal PR. Denoting by H the integral $\int_0^\infty z\psi(z)\, dz$, this action is

$$\pi w^2 \sigma H \left(\frac{1}{R_1} + \frac{1}{R_2}\right), \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and hence equating $\varpi\sigma$ to the sum of (11) and the force $K\sigma$, we have

$$\varpi = K + \pi w^2 H \left(\frac{1}{R_1} + \frac{1}{R_2}\right), \quad . \quad . \quad . \quad . \quad (12)$$

which is the form obtained by Laplace for the molecular pressure-intensity at all points in the liquid below P'. (See the *Mécanique Céleste*, Supplement to Book X.)

We have supposed the surface of the liquid at P to be concave towards the liquid. If it is convex,

$$\varpi = K - \pi w^2 H \left(\frac{1}{R_1} + \frac{1}{R_2}\right). \quad . \quad . \quad . \quad . \quad (13)$$

Hence we have the following obvious consequences.

1. If a liquid is acted upon by molecular forces only (no *external* forces) the quantity $\frac{1}{R_1} + \frac{1}{R_2}$ must be constant at all points of its bounding surface; for, otherwise we should obtain conflicting values for the intensity of molecular pressure at one and the same point in the body of the liquid.

2. The molecular pressure at a point strictly *on* its bounding surface is zero; for on the portion of liquid contained within the canal PP' and included between P

and a point infinitely close to P the resultant force exerted by the fluid is infinitely small (since the mass contained is infinitely small).

3. The value of the intensity of molecular pressure at a point within the body of a liquid is not a constant related solely to its substance; it depends on the curvature of its bounding surface. If this surface is plane, the intensity is K.

4. If (owing, as we shall see, to the action of *external* forces) it happens that some parts of the bounding surface are plane, others are curved and have their concave sides turned towards the liquid, and others again have their convex sides towards the liquid, the intensity of molecular pressure just below the second kind of points is *greater* and below the third *less* than it is at the plane portions.

(We shall presently see how this is verified in the rise or fall of liquid in capillary tubes when gravity is the external force.)

The constant K can be easily expressed in terms of the function ψ thus: let CD, Fig. 29, represent the plane surface of a liquid, the liquid lying below CD; at any point A on the surface take an infinitely small element of area, σ, and describe on it a normal cylinder or canal, AMR, extending into the liquid indefinitely; then $K\sigma$ is equal to the whole force produced on the liquid within this canal by the whole body of liquid below CD.

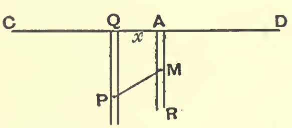

Fig. 29.

Take any point Q on CD; let $AQ = x$, and take the circular strip $2\pi x dx$ round A; along this strip describe normal canals (represented by QP) extending indefinitely into the liquid; take any point, P, in one of these canals;

let $PQ = y$; take any point, M, in the canal at A, and let $AM = z$. Then if s = area of normal section of the canal QP, the element of mass at P is $swdy$, and its action on a mass m at M is $mswdyf(r)$, where $r = MP$; the component of this along MR is $mswdyf(r) . \frac{y-z}{r}$, or $mswf(r)\,dr$; therefore taking the points P at a constant distance y from CD all round the strip $2\pi x dx$, we see that their action on m is

$$2\pi wmxdx . f(r)\,dr.$$

Integrating this from $r = MQ$ to $r = \infty$, we have (putting $MQ = r_1$)

$$2\pi wmxdx . \phi(r_1).$$

Now we must integrate with respect to x from 0 to ∞, and observe that $xdx = r_1\,dr_1$, so that the limits of r_1 are MA (or z) and ∞. Thus we get

$$2\pi wm \int_z^\infty r_1\,\phi(r_1)\,dr_1,$$

$$\text{i.e.,}\quad 2\pi wm\,\psi(z),$$

which is therefore the action of the whole mass of liquid on the particle at M.

Also $m = w\sigma dz$; therefore the action on the canal AR is

$$2\pi w^2\sigma \int_0^\infty \psi(z)\,dz,$$

$$\therefore\quad K = 2\pi w^2 \int_0^\infty \psi(z)\,dz.$$

This constant K is called by Lord Rayleigh the *intrinsic pressure* of the liquid, *Philosophical Magazine*, October, 1890.

72. Pressure on Immersed Area. Suppose a vessel to contain a heavy homogeneous liquid of specific weight w, and let P be a point in the liquid at a depth z below the plane portion of the free surface. Then, as has been shown in the earlier portion of this work, the intensity of pressure at P due to gravitation is wz; and, as has just been proved, the intensity of pressure at P due to molecular forces is K, the intrinsic pressure. Hence the total intensity of pressure is

$$wz + K.$$

Now it is known that K is enormously great: Young estimated its value for water at 23000 atmospheres, while Lord Rayleigh (*Phil. Mag.*, Dec., 1890) mentions, with approval, a hypothesis of Dupré which leads to the value 25000 atmospheres for water. The hypothesis is that the value of K is deducible from the dynamical equivalent of the latent heat of water; that evaporation may be regarded as a process in which the cohesive forces of the liquid are overcome. Now the heat rendered latent in the evaporation of one gramme of water = 600 gramme-degrees (about), or $600 \times 42 \times 10^6$ ergs, and 1 atmosphere = 10^6 dynes per square centimetre; hence $K = 25000$ atmospheres (about).

If this is so, the question must naturally present itself to the student: what becomes of our ordinary expressions for the liquid pressure exerted on one side of an immersed plane area? Instead of being merely $A\bar{z}w$, must it not be very vastly greater—in fact $A(\bar{z}w + K)$? And moreover it should always act practically at the centre of gravity of the area.

We shall see, however, by considering closely the nature of molecular forces, that this large pressure does not influence in any way the value of the pressure exerted by a liquid on the surface of an immersed body. Let us revert to Fig. 28 and consider the result arrived at in Art. 71. This result may be stated thus: at all points on the surface which terminates a liquid—whether this be a free surface or a surface of contact with any foreign body —there is a resultant force intensity due to molecular actions ; this diminishes rapidly as we travel inwards along the normal to the surface, and vanishes after a certain depth has been reached.

If we consider a slender normal canal of any length, PR (Fig. 28) at the boundary of a liquid, this canal will experience *from the surrounding liquid itself* a resultant force acting inwards along its axis PR ; this force is due to molecular actions and the imperfect surrounding of points near the liquid boundary (as explained in Art. 71), and its effects are felt along only a very small length PP' (or ϵ) of the canal. If the boundary of the liquid at I is plane, and the canal has a cross-section σ, this resultant inward molecular force on the canal is $K\sigma$.

Now let Fig. 30 represent a vertical plane, AB, immersed in a liquid having a portion, at least, of its surface, LM, horizontal, and let us consider the pressure exerted per unit area on this plane AB at P at the left-hand side. At P take an infinitesimal element of area, σ, and on it describe a horizontal canal of any length PQ, closed by a vertical area at Q. Consider the equilibrium of the liquid in this canal.

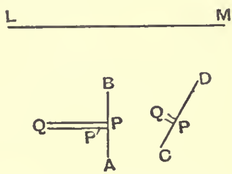

Fig. 30.

Now since AB is a foreign body, there is a termination to the liquid along the surface AB, and hence there will be resultant molecular force exerted by the liquid at points on and very near AB. Hence if along the canal PQ we take the length $PP' = \epsilon$, the liquid in PQ experiences a resultant molecular force from the surrounding liquid, of magnitude $K \,.\, \sigma$, this force acting from P towards Q and being confined to the length PP'. In addition, the solid plane

AB exerts a certain attraction, $a \cdot \sigma$, on the liquid in the canal, together with a certain pressure, $q \cdot \sigma$, on it. Finally, at *Q* the canal experiences the pressure $(wz + K)\,\sigma$ from the liquid. Hence we have

$$(wz + K)\,\sigma = K\sigma + (q - a)\,\sigma,$$

$$\therefore\ q - a = wz,$$

which shows that *K* disappears. Now the action of the canal on the plane at *P* is exactly $(q - a)\,\sigma$ in the sense *QP*, and this action is that which is described in ordinary language as the pressure on the plane at *P*.

If the immersed plane is inclined, as at *CD*, the resultant action of the liquid on the plane at *P* on the element of surface σ is seen in the same way to be $wz\sigma$, by considering the equilibrium of a canal *PQ* normal to the plane, *PQ* being equal to ϵ, the radius of molecular activity.

Laplace is somewhat obscure on the subject of the action between a liquid and an immersed plane (see Supplement to Book X, *Mécanique Céleste*, p. 41). Thus he says: the action experienced by the liquid in the canal *PQ* is equal, 1°, to the action of the fluid on this canal, and this action is equal to *K*; 2°, to the action of the plane on the canal; 'but this action is destroyed by the attraction of the fluid on the plane, and there cannot result from it in the plane any tendency to move; for, in considering only reciprocal attractions, the fluid and the plane would be at rest, action being equal and opposite to reaction; these attractions can produce only an adherence of the plane to the fluid, and we can here make abstraction of them.' He is considering the action experienced by the canal at the extremity *P* where it touches the plane. But, in considering the forces exerted *on the fluid* by the plane, it does not seem allowable to balance any force exerted by the plane on the canal by an opposite force produced *on the plane* by the fluid.

According to the view which we have taken, the action which is commonly called the fluid pressure on the plane is, in reality, a difference action—the difference between a pressure proper and a molecular attraction between the fluid and the plane.

73. Liquid in contact with a Solid. Admitting the existence of molecular forces operative within infinitesimal distances, the surface of a liquid near its place of contact with a solid body must, in general, be curved, even when gravity is the only external force acting throughout the mass of the liquid.

For, let PAB, Fig. 31, represent the surface of a liquid in contact at P with the surface PQ of a solid body.

Consider the forces acting on a molecule at P. We have gravity in the vertical direction Pw; also the molecular forces exerted by the solid evidently produce a resultant along Pn, the normal to the solid at P; and the molecular forces of the fluid molecules adjacent to P produce a resultant, Pf, acting somewhere between the tangent plane to the liquid surface at P and the surface of the solid.

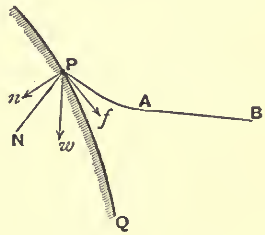

Fig. 31.

Now in all cases the resultant force, due to all causes, exerted on a molecule of a perfect fluid at its free surface must be normal to the surface. Hence the resultant of the forces Pn, Pw, and Pf will determine the normal to the fluid surface at P; and, in general, this resultant will not act along Pw, so that the surface of the fluid at P is not, in general, horizontal.

The form of the surface remote from the solid body is, of course, that of a horizontal plane; because at such points as A and B there are only two forces acting, viz., gravity and the molecular attraction, the latter of which is normal to the surface, and if the resultant of it and gravity is also normal, the force of gravity must act in the normal, i. e., the surface must be horizontal.

74. Application of Virtual Work. When, under the action of any forces whatever, a system of particles assumes a configuration of equilibrium, this configuration is signalized by the fact that if it receives, or is imagined to receive, any small disturbance whatever, the total amount of work done by all the forces acting on the various particles is zero.

We shall now apply this principle to the equilibrium of a liquid contained within an envelope ACB (Fig. 32), the surface of the liquid being APB, and the forces acting being—

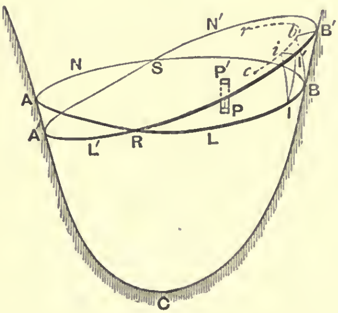

Fig. 32.

1°. molecular forces between particle and particle of the liquid,

2°. molecular forces between the envelope and the liquid,

3°. Any assigned system of external force.

Let m, m' denote indefinitely small elements of mass of the liquid at a distance r, and assume the force between them to be

$$mm'f(r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let μ denote an element of mass of the envelope, and m any element of mass of the liquid very close to μ, and assume the force between m and μ to be

$$m\mu F(r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The value of r in (1) must be $< \epsilon$, otherwise the force between the elements of mass would be zero; and r in (2) must be $< \epsilon'$, the radius of molecular activity for the solid and the fluid.

The virtual work of the force (1) is $-mm'f(r)\,dr$. Now if, as in Art. 69, we put

$$\int_r^{\epsilon} f(r)\,dr = \phi(r),$$

$$\int_r^{\epsilon'} F(r)\,dr = \psi(r),$$

the total work done by the molecular forces for any system of small displacements is

$$\tfrac{1}{2}\delta\,\Sigma\, mm'\,\phi(r) + \delta\,\Sigma\, m\,\mu\,\psi(r), \quad . \quad . \quad . \quad . \quad (3)$$

the summations extending to all pairs of elements between which the molecular force is exerted, and one-half of the result of the summation relating to pairs of liquid elements being taken, because this summation will bring in each term twice.

If V is the potential, per unit mass, at any point of the liquid where the element of mass dm is taken, the virtual work of the external forces is

$$\delta \int V\,dm. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence the equation of virtual work for any system of displacements of the liquid elements is

$$\delta\left[\int V\,dm + \tfrac{1}{2}\Sigma\, mm'\,\phi(r) + \Sigma m\mu\psi(r)\right] = 0. \quad . \quad . \quad (5)$$

It will be necessary, therefore, to calculate the functions

$$\Sigma mm'\,\phi(r) \text{ and } \Sigma m\mu\psi(r).$$

Since if we take any one element m of the liquid and perform the summation $\Sigma m'\,\phi(r)$ round it, the process is confined within the sphere of radius ϵ having m for centre, we may obviously put

$$\Sigma\, m'\,\phi(r) = L, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

L being a constant throughout the whole of the fluid contained in the vessel and bounded—not by the surface

APB but—by the parallel surface $A'B'$ (see Fig. 28) which is at the constant distance ϵ below *APB*, and also by a surface inside the fluid parallel to that of the vessel at a distance ϵ from the surface of the vessel; for each element m within this space is completely surrounded by a liquid sphere of radius ϵ, while the liquid elements between the surfaces *AB* and $A'B'$, and between the vessel and the second surface named are not completely surrounded. Or, if we please, we may imagine the summation L to extend up to the bounding surface *AB* and to that of the vessel, and subtract a summation relating to a fictitious layer, $A''B''$, above *AB* of constant thickness ϵ, included between *AB* and $A''B''$ (Fig. 33), and a fictitious layer outside the surface of contact with the vessel, also of thickness ϵ.

Hence if M is the whole mass of the liquid, the summation can be expressed in the form

$$M \,.\, L - \sigma\, mm' \phi(r), \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which σ denotes a summation confined within the superficial layer, which is everywhere of the constant thickness ϵ, and which embraces the free surface and the surface of contact of the liquid with the vessel.

As regards the summation $\Sigma m \mu \psi(r)$, it is obviously confined between two surfaces each of which is parallel to the surface, *ACB*, one inside the liquid and the other inside the solid envelope, the distance between these surfaces being $2\epsilon'$.

Hence equation (5) becomes

$$\delta \left[\int V dm - \tfrac{1}{2} \sigma\, mm' \phi(r) + \Sigma m \mu \psi(r)\right] = 0. \quad . \quad (8)$$

Now we can easily see that the summation σ is proportional to the sum of the areas of the surface *AB* and that of contact with the vessel; for, if we draw any surface, *XY*, Fig. 33, parallel to *AB* and $A''B''$ within the fictitious surface layer above indicated, and at any point Q on *XY* take the element dm of mass, describing round Q a sphere of radius ϵ, the summation $dm \,.\, \sigma\, m' \phi(r)$ will extend to the volume of the sphere included between AB and $A''B''$; and if z is the normal distance, Qp, of Q from AB, the summation $\sigma\, m' \phi(r)$ can obviously be written $w\,\Pi(z)$; also if dS is a small element of area of XY at Q, we can take

$$dm = w\,dS\,dz,$$

where w is the mass per unit volume of the fluid; hence we have for this element of the fluid the term

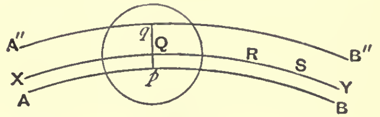

Fig. 33.

$$w^2 dS \,.\, \Pi(z)\, dz.$$

Now we can make a summation from p to q along a cylinder whose cross-section is everywhere dS if the radii of curvature of the surfaces AB, XY, $A''B''$ are infinitely greater than ϵ. The result of this summation is

$$w^2 dS \int_0^{\epsilon} \Pi(z)\, dz,$$

the definite integral being the same at all points, p, of AB. If the definite integral is denoted by A, and we then sum the results all over AB, we have AwS, where S is the area of AB; and similarly for the part of σ which extends over the surface of the vessel.

In the same way it is obvious that the term $\Sigma m \mu \psi(r)$ is proportional to the product of the densities of the envelope and the liquid and to the area, Ω, of their surface of contact. We may therefore write (5) in the form

$$\delta\left[\int V dm - \frac{k}{2}\cdot(S+\Omega) + \lambda\,.\,\Omega\right] = 0, \quad . \quad . \quad . \quad (9)$$

where k and λ are constants which depend on the densities of the liquid and solid and on the laws of intermolecular action.

A result of the same form will obviously hold if the density of the liquid varies both at the surface AB and at the surface of contact with the envelope, provided that the thickness of the stratum of variable density near AB is everywhere the same, and the same at all points along each surface, XY, parallel to AB; and similarly for the stratum near the envelope.

Hence, then, the work done by the molecular forces for any imagined displacement is entirely superficial, and its two parts are proportional to the small increments of the area of the surface AB of the liquid and the surface of contact with the envelope.

We shall now calculate δS and $\delta \Omega$.

In Fig. 32 the new surface, $A'L'I'B'N'$, of the fluid (resulting from the imagined small disturbance of the fluid) can be considered as consisting of two parts: firstly, the portion bounded by the curve $cibr$ ... which is formed by the feet of the normals to this new surface or to the old one (since they differ infinitely little in position) drawn at all the points A, L, I, B, ... of the contour, $ALIBN$; and secondly, the small strip included between the contour $A'L'B'N'$ and the curve $cibr$ ...; so that δS is the area of this strip plus the excess of the first of these portions over the area of the old surface of the fluid. Also $\delta \Omega$ is represented in the figure by the surface $BILRI'B'N'SB$ minus the surface $ARL'A'SNA$, each of these lying on the interior of the vessel.

A simple geometrical investigation of the first part of δS is as follows: at any point P (Fig. 32) on the old surface of the fluid draw the two *principal* normal sections, PQ, PJ (Fig. 34) of the surface, and the normal, $PC_1 C_2$, to the surface at P; take the element of area, $PQFJ$, determined by the elements $PJ = ds_1$ and $PQ = ds_2$; this is an infinitesimal rectangle since ds_1 and ds_2 are at right angles.

Let C_1 and C_2 be the centres of curvature of the principal sections, and

$$PC_1 = R_1,\ PC_2 = R_2,$$

the radii of curvature of these sections.

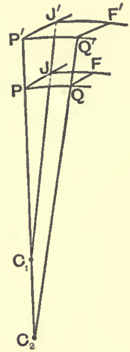

Fig. 34.

Produce the normals to the surface at the points $PQFJ$ to meet the new surface of the fluid in $P'Q'F'J'$, and denote PP' by δn. Then we determine the small rectangular area $P'Q'F'J'$ on the new surface, and the excess of this above $PQFJ$ when integrated over the whole of the old surface is the first part of δS. The figure assumes that the *concave* side of the surface is turned towards the liquid.

Now

$$P'J' = \left(1 + \frac{\delta n}{R_1}\right) PJ,$$

$$P'Q' = \left(1 + \frac{\delta n}{R_2}\right) PQ,$$

therefore if dS = area $PQFJ$, and $dS + \delta dS$ = area $P'Q'F'J'$, we have

$$\delta dS = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \delta n\, dS.$$

Hence the first part of δS is

$$\int \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \delta n\, dS. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

To find the second part of δS, i.e., the sum of all such elements as $iI'B'b$, let $d\omega$ be the element of area $II'B'B$ of the interior of the vessel, and let θ be the angle between the tangent plane to the surface of the liquid at B and the tangent plane to the surface of the vessel at B, i.e., the angle between the plane $iI'B'b$ and the plane $II'B'B$; then

$$\text{area } iI'B'b = \cos\theta \,.\, d\omega. \quad \ldots \quad (11)$$

Hence the second part of δS is

$$\int \cos\theta \,.\, d\omega \quad \ldots \quad (12)$$

taken all round the curve of contact of the fluid and the vessel.

Then, we have

$$\delta S = \int\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\delta n dS + \int \cos\theta d\omega\,; \quad (13)$$

and (9) becomes

$$\delta\int V dm - \frac{k}{2}\int\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\delta n dS + \frac{1}{2}\int(2\lambda - k - k\cos\theta)\,d\omega = 0, \quad (14)$$

since $\delta\Omega$ is obviously the integral of all such elements as $II'B'B$, i.e., $\int d\omega$.

Now observe, however, that (14) is the equation of Virtual Work irrespective of the condition that the volume of the fluid remains the same after displacement as before. The excess of the new volume over the old is obviously the sum of such prismatic elements of volume as that contained between the areas $PQFJ$ and $P'Q'F'J'$ (Fig. 34) whose expression is $\delta n dS$, added to the sum all round the curve $ARLIBNA$ of such wedge elements as $IibB'BI'$. The volume of this wedge is $\frac{1}{2}\delta n_0 \,.\, \cos\theta\, d\omega$, if δn_0 denotes Ii, the normal distance between the new and the old surface of the fluid at any contour point, I; and hence the sum of the wedges will add nothing to the contour integral

$$\int(2\lambda - k - k\cos\theta)\,d\omega$$

in (14), since each element of this sum is an infinitesimal compared with $d\omega$.

Hence, the whole volume of fluid being constant,

$$\int \delta n dS = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

We know from the principles of the Lagrangian method (*Statics*, vol. ii, chap. xv) that the condition of unchanged volume combined with the principle of Virtual Work is expressed by multiplying the left-hand side of (15) by an arbitrary constant, h, and adding it to the left-hand side of (14). Hence, then, the complete equation is

$$\delta \int V dm + \int \left\{ h - \frac{k}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right\} \delta n dS$$
$$+ \frac{1}{2}\int (2\lambda - k - k\cos\theta)\, d\omega = 0. \quad (16)$$

We may finally simplify the term $\delta \int V dm$. It means simply the variation of the potential of the external forces due to changed configuration of the liquid; and this variation is due merely to the two wedges $BILRI'B'N'SB$ and $ARL'A'SNA$, being positive for one and negative for the other. The type of the variation is the variation for the element of mass contained in the small prism PP', Fig. 32, that is $w\delta n dS$; so that if V is the potential of the external forces (per unit mass) at any point P on the surface of the liquid, the work of these forces for any small change of configuration is

$$w \int V \delta n dS, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and therefore (16) finally becomes

$$\int \left\{ wV + h - \frac{k}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right\} \delta n dS$$
$$+ \frac{1}{2}\int (2\lambda - k - k\cos\theta)\, d\omega = 0. \quad (18)$$

The first integral is one extended over the surface of the liquid, and the second is one relating only to its contour, i.e., its bounding curve *ALIBSNA*.

Now, owing to the perfectly arbitrary displacement of every point on the surface, each element of the first integral must vanish, and hence at every point of the surface we have

$$wV + h - \frac{k}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = 0, \quad . \quad . \quad . \quad . \quad (19)$$

which is the equation of the surface.

Every element, also, of the contour integral must vanish, and hence at all points of contact of the surface of the liquid with the vessel

$$\cos\theta = \frac{2\lambda - k}{k}, \quad . \quad . \quad . \quad . \quad . \quad (20)$$

which shows that *the liquid surface is inclined at the same angle to the surface of the vessel all round.* The angle θ is called the *angle of contact* of the liquid and the solid, which we shall definitely suppose to be the angle contained between the normal to the liquid surface drawn into the substance of the liquid and the normal to the solid drawn into the substance of the solid.

If $\lambda > k$, the angle of contact is imaginary, and equilibrium of the liquid in the vessel is impossible.

If the *convex* side of the surface is turned towards the liquid, we shall have

$$P'J' = \left(1 - \frac{\delta n}{R_1}\right) PJ,$$

$$P'Q' = \left(1 - \frac{\delta n}{R_2}\right) PQ,$$

$$\therefore \quad \delta dS = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right) dS,$$

and (19) is replaced by

$$wV + h + \frac{k}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = 0. \quad . \quad . \quad . \quad (21)$$

If the density of the liquid is not constant (owing to the variable molecular pressure) in the layers near the surface, it will be the same at all points on a surface parallel to AB (Fig. 33) at a distance $< \epsilon$ from AB; and hence it is obvious that the virtual work of the molecular forces for any small displacement will still be proportional to the variation, δS, of area of the surface, but the value of the constant, k, will not be the same as on the supposition of constant density. The equation of Virtual Work will, then, be still of the form (9), and the results (19) and (20) will still hold.

If above the surface AB there is another fluid, virtual work of its external and molecular forces will give terms of the same form as before, as will be shown in a subsequent article.

If at each point of the free surface of a liquid there is an external pressure whose intensity at a typical point on the surface is p_0, the virtual work of this pressure must be brought into equation (9) or (16). This virtual work is obviously $-\int p_0 \delta n dS$, so that the equation (19) of the free surface becomes

$$wV - p_0 + h + \frac{k}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = 0. \quad . \quad . \quad . \quad (22)$$

From (20) we see that the angle of contact of a liquid with a solid will be $< \frac{\pi}{2}$ if $\lambda > \frac{k}{2}$, i. e., the surface of the liquid will be convex towards the liquid at the place of contact. If the law of attraction between the liquid elements themselves is the same as that of attraction between a liquid element and an element of the solid in contact with it, ϕ and ψ in (5) differ only by constant multipliers; and we can state the last result thus: if the attraction of the solid on the fluid is greater than half the attraction of the fluid on itself, the surface of the fluid will at the curve of contact be convex towards the fluid; and if $\lambda < \frac{k}{2}$, θ will be $> \frac{\pi}{2}$, i.e., the surface of the fluid will be concave towards the fluid. We shall subsequently see that in the case in which a capillary tube dips into a liquid which is under the action of gravity, the liquid must rise in the tube in the first case, and fall in the second. These results were first enunciated by Clairaut.

The experimental determination of the angle of contact of a liquid and a solid has been made by means of the measurement of a large drop, $OCyD$, Fig. 35, of the liquid placed on a horizontal plane, AB, made of the solid. If the drop is a very large one, it is virtually a plane surface at its highest point O. Suppose the figure to represent a vertical section. Then at any point P the two principal sections are the meridian curve PO and the section made by a plane perpendicular to the plane of the paper through the normal to the curve at P. The curvature of this section may be neglected in the case of a large drop; and if ρ is the radius of curvature of the meridian at P, we have from (12) of Art. 71 the intensity of molecular pressure at P equal to $K + \frac{T}{\rho}$, where T is a constant. At O the intensity of molecular pressure is K, and if the depth, Pm, of P below

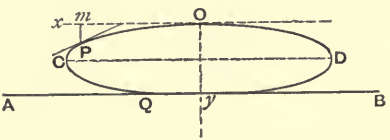

Fig. 35.

the tangent Ox is y, the intensity of pressure at P is also $K+wy$, by transmission from O. Hence

$$wy = \frac{T}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If the arc $OP = s$ and θ is the angle made by the tangent at P with Ox, $\frac{1}{\rho} = \frac{d\theta}{ds}$, and $\sin\theta = \frac{dy}{ds}$; hence (23) becomes

$$wy\,dy = T\sin\theta\,d\theta,$$

$$\therefore\ wy^2 = 2T(1-\cos\theta). \quad . \quad . \quad . \quad . \quad (24)$$

Let the angle of contact at Q be i, let $Oy = a$; then

$$wa^2 = 2T(1+\cos i). \quad . \quad . \quad . \quad . \quad (25)$$

Let CD be the equatorial section of the drop; then at C we have $\theta = \frac{\pi}{2}$, and if the depth, b, of C below O is measured, we have

$$wb^2 = 2T. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

This last gives T, which is called (see Art. 73) the surface tension of the liquid in contact with air; and then (25) gives

$$\cos i = \frac{a^2}{b^2} - 1, \quad . \quad . \quad . \quad . \quad . \quad (27)$$

which determines the angle of contact.

The above arrangement is suitable in the case of a drop of mercury.

To find the angle of contact between water and any solid body, a somewhat similar method has been employed. Imagine Fig. 35 to be inverted, and suppose AB to be the horizontal surface of a mass of water (which then occupies the lower part of the figure). Along this surface fix a plane of the given substance, and under this plane insert a large bubble of air, $QCOD$, the lowest point of the bubble being O.

Then, exactly as before, by measuring the thickness, Oy, of the bubble, and the depth of C below AB, we obtain the angle of contact at Q between the water surface and that of the solid (the water surface being bounded by air). If $a = Oy$, and $b =$ the vertical height of C above O, we have, as before

$$\tfrac{1}{2} wb^2 = T,$$
$$\tfrac{1}{2} wa^2 = T(1 + \cos i),$$

i being the angle of contact CQA.

It is very difficult, if not impossible, to find a definite value of the angle of contact between a given liquid and a given solid, because any contamination or alteration of either surface during the experiment will affect the result. Thus, the angle of contact between water and glass is often said to be zero, while some experimenters quote it at 26°. Again, it is known that in the case of mercury and glass the angle varies with the time during which they are in contact: at the beginning of an experiment the angle was found to be 143° and some hours afterwards 129°.

75. Analytical investigation of general case. The expression (13) of Art. 74 can be analytically deduced from the general theory of the displacements of points on any unclosed surface. Thus if, as in *Statics*, vol. ii, Art. 291, we denote the components of displacement of any point (x, y, z) by u, v, w, and if at any point, P, Fig. 32, on the surface we put $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$, $\epsilon = \sqrt{1 + p^2 + q^2}$, we know that the change, δdS, of the infinitesimal area dS at P is given by the equation

$$\delta dS = \left\{ \epsilon \left(\frac{du}{dx} + \frac{dv}{dy} \right) + \frac{p\,\delta p + q\,\delta q}{\epsilon} \right\} dx\,dy. \quad . \quad . \quad (1)$$

Also (*Statics*, Art. 283)

$$\delta p = \frac{dw}{dx} - p\frac{du}{dx} - q\frac{dv}{dx}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\delta q = \frac{dw}{dy} - p\frac{du}{dy} - q\frac{dv}{dy} \cdot \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence

$$\delta S = \iint \frac{1}{\epsilon}\left\{(1+q^2)\frac{du}{dx} + (1+p^2)\frac{dv}{dy} - pq\left(\frac{dv}{dx} + \frac{du}{dy}\right) + p\frac{dw}{dx} + q\frac{dw}{dy}\right\} dx\,dy. \quad (4)$$

Now, by the method of integration by parts, we have the double integral equal to

$$\int \frac{1}{\epsilon}\{(1+q^2)\,u\,dy - pq\,u\,dx + (1+p^2)\,v\,dx - pq\,v\,dy + pw\,dy + qw\,dx\}$$

$$-\iint\left\{u\left(\frac{d}{dx}\frac{1+q^2}{\epsilon} - \frac{d}{dy}\frac{pq}{\epsilon}\right) + v\left(\frac{d}{dy}\frac{1+p^2}{\epsilon} - \frac{d}{dx}\frac{pq}{\epsilon}\right) + w\left(\frac{d}{dx}\frac{p}{\epsilon} + \frac{d}{dy}\frac{q}{\epsilon}\right)\right\} dx\,dy, \quad (5)$$

in which, of course, the single integral is one carried along the bounding curve, *ALBN* (Fig. 32) of the surface, while the double integral is one carried over the surface itself.

Dealing with the double integral in (5) first, we easily find that the coefficient of u is

$$-p\frac{(1+p^2)\,t + (1+q^2)\,r - 2pqs}{\epsilon^3}, \quad . \quad . \quad . \quad (6)$$

where $r = \frac{dp}{dx}$, $t = \frac{dq}{dy}$, $s = \frac{dp}{dy}$.

But (Salmon's *Geometry of Three Dimensions*, chap. xi) the multiplier of $-p$ in (6) is $\frac{1}{R_1} + \frac{1}{R_2}$. Similarly the coefficient of v in the double integral is $-q\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$, and that of w is $\frac{1}{R_1} + \frac{1}{R_2}$; so that the double integral in (5) is

$$\iint\left(\frac{1}{R_1} + \frac{1}{R_2}\right)(pu + qv - w)\, dx\, dy. \quad . \quad . \quad . \quad (7)$$

But, u, v, w being the components of displacement of the point P, and $\frac{p, q, -1}{\epsilon}$ the direction-cosines of the normal to the surface at P, the projection of the displacement of P along the normal, which we have called δn, is given by

$$\delta n = \frac{pu + qv - w}{\epsilon} dx\, dy$$

$$= (pu + qv - m)\, dS.$$

Hence (7) is

$$\iint\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\delta n\, dS,$$

as before found.

Dealing now with the single integral in (5), and carrying it *continuously* round the bounding curve, we see that the sign of every term in dx must be changed, as is fully explained in *Statics*, Art. 316*a*.

Of course this integral is one which we may consider as carried round the projection on the plane xy of the bounding curve. Hence the correct form of the single integral is

$$\int\frac{1}{\epsilon}\{(1 + q^2)\, u\, dy + pqu\, dx - (1 + p^2)\, v\, dx - pq\, v\, dy$$
$$+ pw\, dy - qw\, dx\}, \quad (8)$$

in which u, v, w are components of displacement of the points on the surface, Ω, of the solid body where it is intersected by the surface of the liquid; so that if $\frac{p', q', -1}{\epsilon'}$ are the direction-cosines of the normal to Ω at any point, I, we have now

$$w = p'u + q'v.$$

Also, the projection of the element $d\omega$ of area of the surface of the vessel at I on the plane of xy is $udy - vdx$, so that

$$d\omega = \epsilon'(udy - vdx).$$

And since dx, dy, dz are proportional to the direction-cosines of the element IB of the bounding curve which is at right angles both to the normal to the liquid and to the normal to the solid, we have $pdx + qdy - dz = 0$ and $p'dx + q'dy - dz = 0$, from which

$$(p - p')\,dx + (q - q')\,dy = 0.$$

Hence the terms multiplying $\frac{1}{\epsilon}$ in (8) are equivalent to $(pp' + qq' + 1)(udy - vdx)$, i.e., to $\epsilon \cos\theta\, d\omega$; so that (8) is simply

$$\int \cos\theta\, d\omega,$$

as was found in (12), Art. 74.

(The alteration of signs in the terms of the single integral in (5) which is rendered necessary by the carrying of the integral by continuous motion in the same sense round the bounding curve *ALBN*, or its projection on the plane xy, is a circumstance which, perhaps, the student would be very likely to overlook.)

76. Liquid under the action of Gravity. In the particular case in which gravity is the only external force acting on a liquid which has air or any other gas above its surface, if z is the height of any point on the surface above a fixed horizontal plane, $V = -z$, and as w' is negligible compared with w, the equation (19) of the surface of the liquid is

$$wz + \frac{k}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \text{Constant}, \quad . \quad . \quad . \quad (1)$$

from which it follows that if there are any points on the surface at which the concavity is turned towards the liquid, i. e., $\frac{1}{R_1} + \frac{1}{R_2}$ is positive, those points must be at a lower level than the points at which the surface is plane; and points where the surface is convex towards the liquid must be at a higher level than the plane portions.

Thus, supposing that two cylindrical capillary tubes, *BC*, *FE*, immersed vertically in a given liquid, and of such different materials that the surface of the liquid in one is concave, and in the other convex, towards the liquid, if L is the plane from which z is measured and if R, R' are the radii of curvature of the surfaces of the liquid within the tubes at C and F, and ζ, ζ' the heights of these points above L, we must have

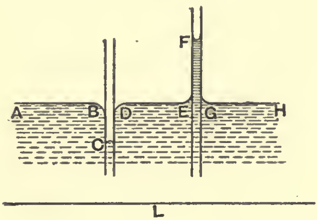

Fig. 36.

$$w\zeta + \frac{k}{R} = w\zeta' - \frac{k}{R'} = wz,$$

where z is the height of the plane portion, *AB*, *DE*, *GH*, of the liquid above L.

A simple experiment with water serves to illustrate the result that if in a continuous body of this liquid there is a part of the surface plane and another convex towards the liquid, this latter must be at a higher level than the former.

Let a large glass vessel be connected with a capillary tube, *t*, Fig. 37, and let water be poured continuously into the large branch. A stage will be reached at which the water in *t* will just reach the top of this tube, and then the surface of the water is *acb*. If the glass is quite clean, this surface will have its tangent planes vertical round the rim *ab*; and the level of the water in the large branch will be at *C*, which is lower than *ab*. As water continues to be poured in, the surface in *t* will become quite horizontal and represented by the right line *ab*, and the surface in the other branch will be *AB*, which is at the same level as *ab*. By continuing to pour in water, the surface at the top of *t* will become *concave* downwards, as represented by *adb*, and then the level in the other branch is at *D*, which is higher than *d*.

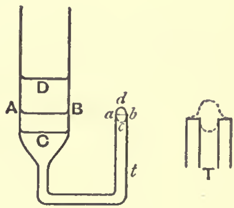

Fig. 37.

A side-figure at *T* shows how the surface of the liquid can be *adb*. The horizontal edges at the top of *t* are of appreciable breadth, and when the water rises above the line *ab*, the surface of the glass is the horizontal rim of the tube *t* (and the angle of contact being 0°) the surface of the water at the rim lies horizontally.

77. Rise or fall of Liquids in Capillary Tubes. It is a well-known fact that if a tube of very small diameter is plunged into a mass of liquid contained in a vessel, the level of the liquid in the vessel will not, in general, be the same as its level in the narrow tube. What is the cause of this? To say that it is 'capillary attraction' is to use an expression which is at once inaccurate and vague. To say that it is molecular attraction is to use an expression which is true but vague. This was evidently in the mind of

Laplace when he said (Supplement to Book X, p. 5) that the attraction of capillary tubes has no influence on the elevation or depression of the fluids which they contain, except in determining the inclination of the surface of the fluid to the surface of the tube along the curve of intersection of the free surface with the tube, and thereby determining the curvature of the free surface. That the angle of contact does determine the curvature of the surface when this surface is inside a very narrow tube is obvious.

For, suppose that the angle of contact for glass and a certain liquid is 45°, and that the liquid is contained in a vertical glass tube one-tenth of a millimetre in diameter; then it is evident that the free surface of the liquid within the tube must be very much curved, because its tangent planes where it meets the tube must all be inclined at 45° to the vertical, while its tangent plane at its vertex must be horizontal; and in order that such a great amount of change in the direction of the tangent plane should be possible, the surface must be very much curved.

Now, great curvature of surface means great intensity of molecular pressure, if the surface is concave towards the liquid, and small intensity if the surface is convex towards the liquid (Art. 71).

Hence, owing merely to the fact that within a very narrow tube, the free surface of a liquid is curved—and not to any special action due to the *narrowness* of the tube—this liquid must rise or fall within the tube below the level of the plane portions in any vessel into which the narrow tube dips.

Let *FE*, Fig. 36, be a capillary tube dipping into a vessel containing a liquid such that the angle of contact (as defined in p. 134) for the liquid and the tube is $< \frac{\pi}{2}$.

In this case the surface is concave upwards, and therefore the intensity of molecular pressure at F is $K - \frac{k}{R}$, where R is the radius of curvature of the liquid surface at the lowest point of the surface at F (where the two radii R_1, R_2 are evidently equal), and the liquid must rise in the tube until the intensity of pressure due to the weight of the column FE added to this molecular intensity produces the intensity of pressure which exists along DE. If p_0 is the intensity of atmospheric pressure, and $EF = z$, the intensity of pressure inside the tube at the level E is $p_0 + wz + K - \frac{k}{R}$; and the intensity of pressure along the plane surface DE is $p_0 + K$; hence

$$z = \frac{k}{wR} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

determines the height to which the liquid rises in the tube.

Let i be the angle of contact of the liquid with the tube and r = internal radius of tube; then $R = r \sec i$, very nearly; hence

$$z = \frac{k \cos i}{wr}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the weight of the liquid raised in the tube above E is

$$\pi r k \cos i. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equation (2) shows that the heights to which the same liquid rises in capillary tubes of the same substance are inversely proportional to the diameters of the tubes.

If the tube is such that the angle of contact is $> \frac{\pi}{2}$, the free surface within the tube is concave towards the liquid, and therefore the intensity of molecular pressure is greater than at the plane surface *DE*. Hence the liquid must be depressed in the tube, as represented in *BC*, and the amount of depression is calculated as above.

78. Surface Tension of a Liquid. The amount of rise or fall of a liquid, under the action of gravity, in a capillary tube is usually calculated by means of the introduction of the notion of *surface tension*. The free surface of the liquid is considered to be in a state of tension resembling that of a stretched surface of indiarubber—with, however, this important difference, that, whereas the tension of the indiarubber surface increases if the surface is further increased, the tension of the liquid surface remains absolutely constant whether the surface expands or contracts.

Let *ABCD*, Fig. 38, represent a part of the bounding surface of a liquid; let any line *QPR* be traced on it; and along this line draw small lengths *Qq*, *Pp*, *Rr* into the liquid and normal to the surface. Consider now the action exerted over the area *RrpqQP* by the liquid at the right side on that at the left.

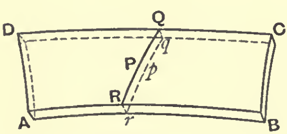

Fig. 38.

One part of this action will consist of molecular attraction, towards the right; and if the depth of the line *qpr* below *QPR* is nearly equal to ϵ (p. 109) or greater than ϵ, another part of the action will consist of pressure, towards the left, in the lower parts of the area *QqrR*. If *qpr* instead of being at an infinitely small depth is at any finite depth, the molecular attraction exerted across any of the lower portions of the area *QqrR* is exactly balanced by the molecular pressure on such portions. But if *qpr* is at a depth *very* much less than ϵ, the molecular pressure (towards the left) on any part of the area *QqrR* is negligible, and we may consider the portion of liquid at the right of the line QR as simply exerting a pull or tension on the portion at the left. Observe that on the line QPR itself the intensity of molecular pressure is zero, and that if qpr is at a distance infinitely less than ϵ from QPR, although the intensity of pressure on any portion of the area $QqrR$ is infinitely small, the force of molecular attraction exerted across this area may be large. The value of the pressure depends on the force by a *differential* relation $\frac{d\varpi}{dn} = wF$ (see (2) of p. 114), where F is the intensity of molecular force in the direction of the normal to the surface of the liquid at any point, P; and we know that at P, where F is greatest, ϖ is zero.

Hence at points of the imagined surface $QqrR$ of separation which are infinitely near to the surface we are to imagine the stress to be merely tension; at points whose distances from the surface become comparable with ϵ we are to imagine this molecular pull, or attraction, as accompanied by a contrary pressure; and at points which are at and beyond the distance ϵ from the surface, the molecular pull is balanced by the molecular pressure.

Hence, however far the imagined surface $QqrR$ extends into the liquid, the whole stress exerted on the liquid at the left by that at the right is confined to an action which terminates at a curve, qpr, at the depth ϵ, this action being a mixed one consisting of molecular attraction and an opposing molecular pressure, which latter grows in intensity from zero at the surface to a maximum value at the depth ϵ.

At any point P on the surface the amount of this stress (which is, on the whole, a tension) *per unit length of the curve PQ is called the surface tension exerted across the curve PQ* by the liquid at the right on that at the left of PQ.

It is obvious that the amount of the stress per unit length is the same across *all* curves drawn at P on the surface, and is normal to these curves. This is another point of difference between the surface tension of a liquid and the stress of an elastic membrane in general; for, in the latter the stress exerted across any curve PQ at P is not, in general, normal to PQ, nor is it of constant magnitude for all curves drawn in the membrane at P.

Now, although it is obvious that, if we grant the existence of molecular forces, we must admit the existence of this mixed surface stress (i. e., within a layer of thickness ϵ at the surface) no such stress has explicitly presented itself in our investigation, by means of Virtual Work, of the conditions of equilibrium of the liquid. This fact, however, involves no difficulty or contradiction; for, in taking the molecular actions exerted between all possible pairs of elements of mass, we are sure of having omitted no forces that act; but in this way surface tension (which is obviously a resultant, and not a simple, action) could not have specially presented itself.

Knowing now of the existence of this stress, we can see why the terms in the expression for the Virtual Work of the molecular forces, (9), p. 129, consist of constants multiplied by the changes of the areas $S+\Omega$, and Ω.

For, if a surface of area A is subject to a tension T which is the same at all points and of constant intensity in all directions round a point, the work done by the stress in an increase δA of the area is

$$-T \,.\, \delta A.$$

A liquid contained in a vessel, or resting as a drop on a table, is sometimes spoken of as having a 'skin' within which the liquid proper is contained.

A drop of water hanging from the end of a tube and ready to fall is spoken of as being contained in an 'elastic bag.' Of course, if the surface of the liquid is oxidized, contaminated by foreign particles of any kind, or in any way rendered different from the liquid below the surface, we may, if we please, say that the pure liquid is contained within a surface which is not pure liquid; but even such a contaminated surface is radically different from an elastic bag, for the magnitude of the tension in a stretched bag increases with the stretch of the bag, whereas the tension of the bounding layer of the liquid does not. In no case—either that of a perfectly pure liquid or that of a liquid with an oxidized or contaminated surface—is there any skin or bag. In the case of a liquid with a pure surface there is no material thing at the surface which there is not everywhere else in the liquid; and we must not imagine that, because we see a drop of water hanging, a globule of mercury lying on a table, or a column of water with a concave surface standing in a capillary tube above the level of the water outside, such conditions require bags for enclosing the liquids or skins by means of which to catch hold of them. We can assure ourselves that molecular forces, with special circumstances near the surface (owing to incomplete surrounding of molecules, &c.), will amply account for all such forms of equilibrium.

The height to which a liquid rises in a capillary tube may be calculated by the introduction of surface tension.

For, in Fig. 39, let the tube $ABB'A'$ have any form (not necessarily cylindrical); let l be the length of the curve of contact of the liquid surface at BB' with the surface of the tube, let T be the surface tension of the liquid, and i the angle of contact with the tube. Then consider the equilibrium of the column in the tube above the level, OCx, of the plane portion outside.

We may suppose the layer of particles round the tube at

B which are in actual contact with the tube as exerting the tension T per unit length of the curve l on the particles just outside them; hence these supply an upward vertical force $Tl \cos i$ on the column $BOO'B'$. If σ is the area of the cross-section of the tube, and $z =$ height of B above O, the weight of the liquid is $w\sigma z$. There is the downward atmospheric pressure, $p_0\sigma$, at B, and an upward pressure at O consisting of $p_0\sigma$ and of a molecular part, $K\sigma$, and there is finally a downward molecular attraction exerted on those particles in the tube which stand on the area σ at OO' and are contained within the distance ϵ from σ.

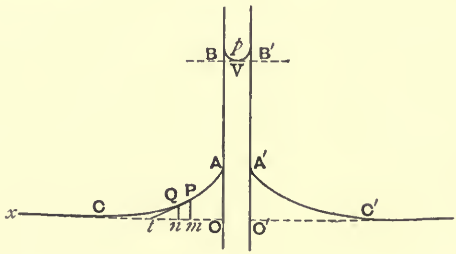

Fig. 39.

Now this downward attraction is precisely equal to $K\sigma$ (Art. 71), so that this force balances the upward pressure $K\sigma$; and we have for the equilibrium of the contained column

$$Tl \cos i = w\sigma z.$$

If the tube is cylindrical,

$$l = 2\pi r, \sigma = \pi r^2,$$

and we have

$$z = \frac{2 T \cos i}{wr},$$

as in (2), p. 145.

It is obvious that $\frac{k}{2}$ in the general equation (9), p. 129, is T, the surface tension.

By taking any element of area of the curved surface BVB', the principal radii of curvature of this element being R_1 and R_2, and considering the equilibrium of the vertical cylinder described round the contour of this element, we at once deduce (1) of p. 142; for, if dS is the area of the element, the component, along its normal, of the surface tension all round dS is $T\left(\frac{1}{R_1}+\frac{1}{R_2}\right)dS$, and the vertical component of this is $T\left(\frac{1}{R_1}+\frac{1}{R_2}\right)d\sigma$, if $d\sigma$ is the horizontal projection of dS; also the weight of the column is $wzd\sigma$; therefore

$$wz - T\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = 0.$$

Let the capillary tube be replaced by two very close parallel vertical plates, AB, $A'B'$.

Then, considering the equilibrium of the column $BOO'B'$ of unit thickness perpendicular to the plane of the figure, we have

$$2T\cos i = wzd,$$

where d is the distance OO' between the plates; hence

$$z = \frac{2T\cos i}{wd},$$

which shows that the liquid rises twice as high in a cylindrical tube as between two parallel plates whose distance is equal to the diameter of the tube.

The existence of surface tension in a liquid may be shown experimentally in many ways, of which we select two. Take a rectangle formed of brass strips or wires, AB, BC, *CD*, and *EF* (Fig. 40), of which the first three form one rigid piece, while the last, *EF*, is capable of sliding up and down on the bars *AB* and *DC*. The space, *abcd*, enclosed by the bars being vacant, dip the system into a solution of soap in water, thus forming a film (represented by the shading) in this space. This film attaches itself to all the bars; and if the movable bar *EF* is not restrained by the hand, it will be drawn along the others by the film until it reaches *BC*. If the system is held in a vertical plane, *EF* being below *BC*, the former will be raised, in opposition to gravity, if it is not too heavy.

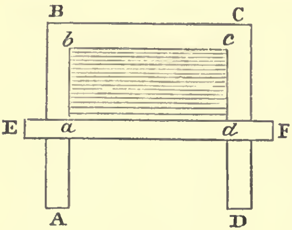

Fig. 40.

As a second example, take a circular brass wire, *A*, Fig. 41, dip it into the soap solution, thus covering its area on withdrawal with a thin film (represented by the shading); then form a loop of a piece of thread and place it gently on the surface of the film. This loop is represented by *ab*. Now perforate the film inside the loop by a pin, and we shall see the loop of thread instantly drawn out into a circle, *c*, by the contracting film. By the formation of this circle the area of the remaining film is the least that it can be—a result which is necessary because, the virtual work of the tension being $T \delta A$, the static energy of the

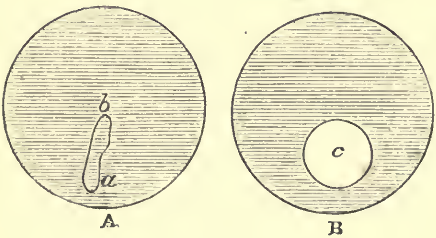

Fig. 41.

film is proportional to its area, and every material system which is subject to given conditions assumes as a configuration of stable equilibrium one in which the static energy of its forces is least.

The following table, taken from Everett's *Units and Physical Constants*, gives the values of a few surface tensions in dynes per linear centimetre at the temperature 20° C.:

	Tension of surface separating the liquid from		
	Air	Water	Mercury
Water	81	0	418
Mercury	540	418	0
Bisulphide of Carbon	32·1	41·75	372·5
Alcohol	25·5	.....	399
Olive Oil	36·9	20·56	335
Petroleum	31·7	27·8	284

from which it is seen how large the surface tension is for mercury as compared with other liquids.

79. Two Liquids and a Solid. To illustrate further the application of the principle of Virtual Work, take the case in which two liquids, w, w' (Fig. 42), are in contact with each other over a surface Λ and with a solid body. The liquid w is contained within the space represented by $ABDC$, and the second within $CDB'A'$. Let S be the area of the free surface of the first, and Ω the area of the solid in contact with this liquid; and S' and Ω' be the corresponding things for the second liquid.

Then, exactly as in Art. 74, the Virtual Work of all the forces acting will reduce to the sum of terms relating to the bounding surfaces alone; and since the whole bounding surface of the liquid w is $S+\Lambda+\Omega$, the virtual work of its own molecular forces will give the term

$$-\frac{k}{2}\delta(S+\Lambda+\Omega);$$

and we see that the equation of virtual work is

$$\delta\int V dm+\delta\int V' dm'-\frac{k}{2}\delta(S+\Lambda+\Omega)-\frac{k'}{2}\delta(S'+\Lambda+\Omega')$$
$$+l\delta\Lambda+\mu\delta\Omega+\mu'\delta\Omega'=0, \quad (1)$$

where the term $l\delta\Lambda$ relates to the molecular forces exerted at the surface Λ between particles of w and particles of w'; and $\mu\delta\Omega$ relates to the forces between the particles of the liquid w and the solid.

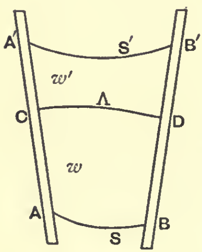

Fig. 42.

Now, denoting by δn, $\delta n'$ elements of normals at points on S, S' drawn outwards from the liquids; $\delta\nu$ an element of normal of Λ drawn outwards from the liquid w; $d\omega$ an element of Ω at the intersection of S and Ω (as in Art. 74), $d\psi$ an element of Ω at the intersection of Ω and Λ; $d\omega'$ an element of Ω' at the intersection of Ω' and S', and by θ, χ, θ' the angles of contact with the solid at S, Λ, S', we have, exactly as shown in Art. 74,

$$\delta\Omega=\int d\omega+\int d\psi,$$

$$\delta\Omega'=\int d\omega'-\int d\psi,$$

$$\delta S = \int \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \delta n dS + \int \cos\theta d\omega,$$

$$\delta S' = \int \left(\frac{1}{R'_1} + \frac{1}{R'_2}\right) \delta n' dS' + \int \cos\theta' d\omega',$$

$$\delta\Lambda = \int \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \delta\nu\, d\Lambda + \int \cos\chi\, d\psi,$$

$$\delta \int V dm = \int V \delta n dS + \int V \delta \nu d\Lambda,$$

$$\delta \int V' dm' = \int V' \delta n' dS' - \int V' \delta \nu d\Lambda,$$

where ρ_1, ρ_2 are the principal radii of curvature at any point on Λ.

To the left-hand side of (1) must be added the terms

$$h \int \delta n dS + h' \int \delta n' dS' + h'' \int \delta \nu d\Lambda,$$

which are rendered necessary by the constancy of the volumes of the liquids in the supposed displacement of the system.

As before, the coefficients of δn, $\delta \nu$, $\delta n'$ must each be zero, as also the coefficients of the terms relating to $d\omega$, $d\omega'$, and $d\psi$. Hence, for example, we have the equation

$$V + h - \frac{k}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = 0$$

at all points of S, and similar equations at all points of S' and Λ; and at all points of meeting of S and Ω

$$\frac{k}{2}(1 + \cos\theta) - \mu = 0,$$

which proves the constancy of the angle of contact at such points; a similar result holding for S' and Ω', while the terms relating to $d\psi$ give

$$(k - k' - 2l)\cos\chi = 2(\mu - \mu') - k - k',$$

which proves the constancy of the angle of contact between the solid and the common surface of the liquids.

80. Drop of Liquid on another Liquid. Let Fig. 43 represent a drop of one liquid resting on the surface of another, the area of contact being Λ, the free surface of the drop being S' and that of the supporting liquid S.

If the sides of the vessel in which this liquid is contained are very distant from the drop, in considering a small deformation of the system and applying the equation (1), Art. 79, of Virtual Work, we may neglect terms relating to Ω; so that the equation is

$$\delta\int Vdm + \delta\int V'dm' - \frac{k}{2}\delta(S+\Lambda) - \frac{k'}{2}\delta(S'+\Lambda) + l\delta\Lambda = 0. \quad (1)$$

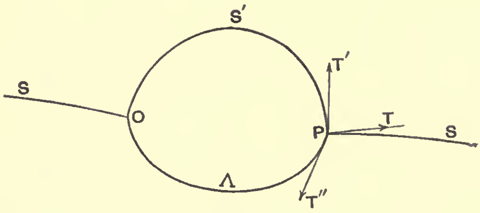

Fig. 43.

This equation will, as has already been seen, give equations satisfied at all points of S, S', Λ, as well as equations relating to their common bounding curve. Considering merely the latter for the present, we may take

$$-\frac{k}{2}\delta(S+\Lambda) - \frac{k'}{2}\delta(S'+\Lambda) + l\delta\Lambda = 0, \quad . \quad (2)$$

in which the variations of the surfaces are only those portions at their common boundary curve; or

$$\frac{k}{2}\delta S + \frac{k'}{2}\delta S' + \left(\frac{k+k'}{2} - l\right)\delta\Lambda = 0, \quad . \quad . \quad (3)$$

or again

$$T\delta S + T'\delta S' + T''\delta\Lambda = 0, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where T, T', T'' are the surface tensions in the surfaces S, S', Λ.

Now when any surface Σ having for bounding edge a curve C receives a very small deformation whereby it becomes a surface Σ' having for bounding edge a curve C', the Calculus of Variations leads (see Arts. 74 and 75) to the result that the variation $\Sigma' - \Sigma$ is obtained by drawing normals to Σ all round the contour C, these normals being terminated by Σ' and enclosing a surface Ω on Σ', and then adding to $\Omega - \Sigma$ a linear integral taken all along the curve C, the elements of this linear integral involving the displacements of points on C to their new positions on C'. The term $\Omega - \Sigma$ is

$$\int \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \delta n \, dS,$$

while the term given by the line-integral along C is (8) of Art. 75.

Now take the case in which the figure of the drop and that of its submerged part are surfaces of revolution round the axis of z, and suppose Fig. 43 to represent the section of it made by the plane xz. Also let the displacement of the point P be confined to the plane xz, and let its components be u, w. Hence in (8) of Art. 75 we are to put $q = 0$, $v = 0$, and the terms of the linear integral which relate to the displacement of P are

$$\frac{1}{\epsilon}(u + pw)\, dy,$$

where dy relates to P and a point on the curve (a circle) which is the common bounding edge of the surfaces S, S', Λ, this circle being represented in projection on the plane of the figure by the right line OP. We may, then, omit dy; and the terms in (4) given by the three surface tensions are

$$\left(\frac{T}{\epsilon} + \frac{T'}{\epsilon'} + \frac{T''}{\epsilon''}\right) u + \left(\frac{Tp}{\epsilon} + \frac{T'p'}{\epsilon'} + \frac{T''p''}{\epsilon''}\right) w.$$

Now u and w are quite arbitrary and independent, and the coefficient of every independent displacement for each point involved in the line-integral must be zero. Hence

$$\frac{T}{\epsilon} + \frac{T'}{\epsilon'} + \frac{T''}{\epsilon''} = 0,$$

$$\frac{Tp}{\epsilon} + \frac{T'p'}{\epsilon'} + \frac{T''p''}{\epsilon''} = 0.$$

But if the tangent line to S in the plane of the figure makes the angle θ with the axis of x, we have

$$-\frac{1}{\epsilon} = \sin\theta \text{ and } \frac{p}{\epsilon} = \cos\theta;$$

similarly for the tangent lines to S' and Λ; so that these become

$$T\sin\theta + T'\sin\theta' + T''\sin\theta'' = 0,$$

$$T\cos\theta + T'\cos\theta' + T''\cos\theta'' = 0,$$

which plainly assert that three forces, T, T', T'', supposed acting along the tangents in the senses represented have no resultant; in other words, if a plane triangle is formed by three lines proportional to the surface tensions, the directions of the distinct surfaces of the two liquids and that of their common surface of contact are parallel to the sides of this triangle.

Hence equilibrium of the drop is impossible unless each surface tension is less than the sum of the other two.

81. Liquid under no external forces. When a mass of liquid is in equilibrium under its own molecular forces only, its surface can assume several forms. In this case (19) of Art. 74 gives as the equation of the surface

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where a is a constant length.

We shall confine our attention to surfaces which are of revolution, and we shall suppose the axis of revolution to be taken as axis of x.

Now if at any point, P, of the revolving curve (Fig. 44) or meridian, PDE, which by revolution round the axis AB generates the surface, ρ is the radius of curvature and n is the length, Pn, of the normal terminated by the axis of revolution, the principal radii of curvature of the surface generated are ρ and n, so that (1) becomes

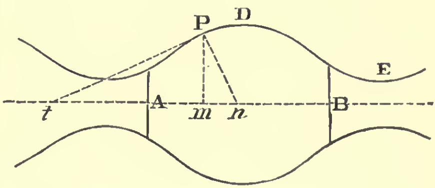

Fig. 44.

$$\frac{1}{\rho}+\frac{1}{n}=\frac{1}{a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now let the tangent at P make the angle θ with the axis of x, and let ds be an element of arc measured along the curve from P towards D; then $\rho=-\frac{ds}{d\theta}$, and $n=y\sec\theta$; hence (2) becomes

$$-\frac{d\theta}{ds}+\frac{\cos\theta}{y}=\frac{1}{a}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or, since $\frac{d\theta}{ds} = \sin\theta \frac{d\theta}{dy}$,

$$\frac{1}{y} \cdot \frac{d}{dy}(y \cos\theta) = \frac{1}{a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\therefore \; y \cos\theta = \frac{y^2}{2a} + h, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where h is a constant.

We may observe that if the constant $\frac{1}{a}$ is zero, (1) gives as the property of the surface of the fluid that at every point the two principal radii are equal and opposite; the two principal sections have their concavities turned in opposite directions. If the surface is one of revolution, this property at once identifies it with the surface generated by the revolution of a catenary round its directrix, and the surface is called a *catenoid*.

Before proceeding to integrate (5), we can show that all the curves satisfying it are generated by causing conic sections to roll, without sliding, along the axis AB: the curves satisfying (5) are the loci traced out by the foci of these rolling conics. For, if $Pn = n$, (5) gives

$$y^2 \left(\frac{1}{n} - \frac{1}{2a}\right) = h. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now if p is the perpendicular from the focus of an ellipse on the tangent at any point, and r the distance of this point from the focus, we have

$$p^2 \left(\frac{1}{r} - \frac{1}{2a}\right) = \frac{b^2}{2a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

a and b being the semiaxes. Comparing (7) with (6), we see that P in Fig. 44 is the focus of an ellipse touching AB at n, the semiaxes being a and $\sqrt{2ah}$, and this ellipse is therefore invariable whatever be the position of P on the meridian. The locus of P when the rolling conic is an ellipse is called the *unduloid*, and is the locus PDE actually represented in the figure.

If the rolling conic is a parabola, the locus of the focus is a catenary, which gives by revolution the catenoid.

If the rolling conic is a hyperbola, the locus is a curve having a series of loops, and the surface which it generates is called a *nodoid*.

Since $\tan\theta = \frac{dy}{dx}$, we have from (5)

$$dx = \pm \frac{y^2 + 2ah}{\sqrt{4a^2y^2 - (y^2+2ah)^2}} \cdot dy \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$= \pm \frac{y^2 + 2ah}{\sqrt{-y^4 + 4a(a-h)y^2 - 4a^2h^2}} \cdot dy \quad . \quad . \quad (9)$$

$$= \pm \frac{y^2 \pm 2a\beta}{\sqrt{(\alpha^2 - y^2)(y^2 - \beta^2)}} \cdot dy, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

by putting $\alpha^2 + \beta^2 = 4a^2 - 4ah$, and $\alpha^2\beta^2 = 4a^2h^2$; so that α and β are the greatest and least values of the ordinate.

Equation (10) is best integrated by expressing y, and therefore x, in terms of a variable angle ϕ; thus, let

$$y^2 = \alpha^2\cos^2\phi + \beta^2\sin^2\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\therefore \quad y = \alpha\sqrt{1 - k^2\sin^2\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $k^2 = \frac{\alpha^2 - \beta^2}{\alpha^2}$. This gives, if $\Delta\phi \equiv \sqrt{1 - k^2\sin^2\phi}$,

$$dx = \left\{ a\Delta(\phi) \pm \frac{\beta}{\Delta(\phi)} \right\} d\phi. \quad . \quad . \quad . \quad . \quad (13)$$

When $\phi = 0$, $y = \alpha$, and when $\phi = \frac{\pi}{2}$, $y = \beta$; so that if D and E are the points of maximum and minimum ordinate, all points between them on the curve are given by values of ϕ between 0 and $\frac{\pi}{2}$.

In the common notation of elliptic integrals

$$\int_0^\phi \Delta(\phi)\, d\phi = E(\phi), \quad \int_0^\phi \frac{d\phi}{\Delta(\phi)} = F(\phi), \quad . \quad . \quad (14)$$

therefore we have the abscissa and ordinate of every point on the curve expressed in terms of the variable ϕ by the equations

$$x = a\,E(\phi) \pm \beta\,F(\phi), \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$y = a\,\Delta(\phi). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

In the unduloid the tangent can never be parallel to the axis of y, i.e., $\frac{dx}{dy}$ can never be zero, so that in (10) the sign $+$ in the numerator belongs to this curve, and therefore in (15) the signs $\pm$ belong, respectively, to the unduloid and the nodoid.

In the unduloid $\frac{d^2x}{dy^2} = 0$ when $\tan\phi = \left(\frac{a}{\beta}\right)^{\frac{1}{2}}$, or $y = \sqrt{a\beta}$, and this gives the point of inflexion on the curve.

If s is the length of the arc between D and any point P, we have $s = (a + \beta)\,\phi$.

When $a = \beta$, the surface generated becomes a cylinder.

When a is very slightly greater than β, the generating curve becomes, approximately, a *curve of sines*.

The case of a liquid unacted upon by any external forces was realized by M. Plateau by inserting a drop of olive oil in a mixture of water and alcohol arranged so as to have the same specific gravity as the oil. By seizing this drop between two wires in the shapes of any closed curves, or by allowing it to form round a solid of any shape held in the water-alcohol mixture, we can obtain a large number of liquid surfaces each satisfying the common equation (1) of such surfaces.

We shall subsequently see that the same surfaces can be produced by means of soap-bubbles instead of large masses of liquids.

A full account of all such experiments will be found in Plateau's celebrated work, *Statique Expérimentale et Théorique des Liquides soumis aux seules Forces Moléculaires.*

82. Liquid under action of Gravity. Taking now the case in which the only external force is gravity, the equation of its surface is of the form

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{z-h}{a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where h and a are constant lengths, and z is the height of any point on the surface above a fixed horizontal plane; also $a^2 = \frac{T}{w}$, where T is the surface tension.

We shall begin by investigating the form of the surface of a liquid in contact with a broad vertical plane, or wall.

Let this plane be supposed normal to the plane of the paper, and let Fig. 39 represent the section of the plane and the liquid surface made by the plane of the paper (supposed also vertical), this section being far removed from the edges of the immersed vertical plane BAO. Of the two principal radii of curvature of the liquid surface at any point P one will be infinite, since one principal section at P is the right line through P perpendicular to the plane of the paper, and the other will be ρ, the radius of curvature of the curve APC.

Taking the axis of x horizontal and the axis of y vertical, we replace z in (1) by y, and the equation becomes

$$\frac{1}{\rho} = \frac{y-h}{a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which shows that the curve *APC* belongs to the class of *elastic curves*, i.e., those formed by a thin elastic rod which when free from strain was straight, but under the action of terminal pressures is bent. (See *Statics*, vol. ii, Art. 306.)

Since at a considerable distance from the wall the surface is plane and $\rho = \infty$, we see that if we measure y from the level of the plane portion, the equation is

$$\rho y = a^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let Ox be the plane level, which is now taken as axis of x.

Putting $p = \frac{dy}{dx}$, which in the figure is − the tangent of the inclination of the tangent at P to the axis of x, we have

$$\frac{1}{\rho} = \frac{p\frac{dp}{dy}}{(1+p^2)^{\frac{3}{2}}}$$

and a first integral of (3) gives

$$\frac{-2}{\sqrt{1+p^2}} = \frac{y^2}{a^2} + C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where C is a constant. Since $p = 0$ when $y = 0$, $C = -2$, and then we have

$$\frac{2a^2 - y^2}{y\sqrt{4a^2 - y^2}} \cdot dy = \pm dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Putting $$y = 2a \sin\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

we have $$\left(\frac{1}{\sin\phi} - 2\sin\phi\right)d\phi = \pm\frac{dx}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now if the angle of contact is acute (as represented in the figure), $\frac{dx}{dy}$ can never vanish, and $\frac{dy}{dx}$ is continuously negative. Hence y is always $< a\sqrt{2}$, and the negative sign in (5) is the one to be used. In this case the integral of (7) is

$$-\frac{x}{a} = 2\cos\phi + \log_e \tan\frac{\phi}{2} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This shows that the plane surface of the liquid is in reality asymptotic to the curve APC, because when $\phi = 0$, $x = \infty$.

If the angle of contact is zero, we have $\frac{dy}{dx} = \infty$ at A, $\therefore y = a\sqrt{2} = OA$.

If i is the angle of contact, $\frac{dy}{dx} = -\cot i$ at A, and we have from (4)

$$OA = a\sqrt{2(1-\sin i)}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$= \sqrt{\frac{2T}{w}(1-\sin i)}, \quad . \quad . \quad . \quad . \quad (10)$$

which determines the height to which the liquid rises against the plate; and, if i is known, by measuring this height the surface tension T can be found.

The equation (3) can be immediately deduced by elementary principles from the notion of surface tension. For, let Q be a point on the curve indefinitely near P; draw the ordinates Pm, Qn, and consider the equilibrium of the prism of liquid $PmnQ$ of unit breadth perpendicular to the plane of the paper. We may consider this prism as kept in equilibrium by the surface tensions, each equal to T, at P and Q, and its weight, the atmospheric pressure cancelling at the top and bottom. Now the vertical upward component of T at P is $T\sin\theta$, and the vertical component at Q is $T\sin\theta + ds \cdot \frac{d}{ds}(T\sin\theta)$; hence

$$-T ds \cdot \frac{d}{ds}(\sin\theta) = wy\,dx,$$

$$\therefore \quad -T\cos\theta\frac{d\theta}{ds} = wy\frac{dx}{ds},$$

$$\therefore \quad \frac{T}{\rho} = wy,$$

since $\rho = -\frac{ds}{d\theta}$, and $\cos\theta = \frac{dx}{ds}$. This equation is the same as (3), since $\frac{T}{w} = a^2$.

The integration of (3) may be effected in another way which gives the intrinsic equation of the curve. It can be written

$$-\frac{d\theta}{ds} = \frac{y}{a^2}, \quad \dots \dots \quad (11)$$

$$\therefore \quad -\frac{d^2\theta}{ds^2} = \frac{1}{a^2}\frac{dy}{ds} = -\frac{1}{a^2}\sin\theta. \quad \dots \quad (12)$$

$$\therefore \quad \left(\frac{d\theta}{ds}\right)^2 = C - \frac{2}{a^2}\cos\theta. \quad \dots \quad (13)$$

Now $-\frac{1}{\rho} = \frac{d\theta}{ds} = 0$ when $\theta = 0$, therefore $C = \frac{2}{a^2}$, and we have

$$\frac{d\theta}{ds} = -\frac{2}{a}\sin\frac{\theta}{2}, \quad \dots \dots \quad (14)$$

$$\therefore \quad \log_e \tan\frac{\theta}{4} = -\frac{s}{a} + C,$$

where C is a constant. Now at A we have $\theta = \frac{\pi}{2} - i$, therefore

$$\tan\frac{\theta}{4} = e^{-\frac{s}{a}}\tan\left(\frac{\pi}{8} - \frac{i}{4}\right) \quad \dots \dots \quad (15)$$

is the intrinsic equation of the curve.

Pass now to the case in which two large parallel vertical plates, BO, $B'O'$ (Fig. 39), are immersed, very close together in a liquid. Let BVB' be the curve in which the liquid surface between them is intersected by the plane of the figure, V being the lowest point, or vertex, of the curve.

One of the principal radii of curvature at every point, p, of this surface is still ∞, and the other is ρ, the radius of curvature of the curve BVB' at the point. Hence, measuring the height of p above the plane surface Ox, we have still

$$\rho y = a^2;$$

and if θ is the inclination of the tangent at p to the horizon and $s = Vp$,

$$\rho = \frac{ds}{d\theta}, \quad \sin\theta = \frac{dy}{ds};$$

therefore

$$\frac{d^2\theta}{ds^2} = \frac{\sin\theta}{a^2},$$

$$\therefore \quad \frac{d\theta}{ds} = \frac{\sqrt{2}}{a}\sqrt{C - \cos\theta}, \quad . \quad . \quad . \quad (16)$$

where C is a constant. Hence

$$y = a\sqrt{2}\,\sqrt{C - \cos\theta}; \quad . \quad . \quad . \quad . \quad (17)$$

and if h is the height of V above OO',

$$h = a\sqrt{2}\,\sqrt{C - 1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

An approximate value of h has been already found (p. 151). If the abscissa of p with reference to V as origin is x, we have $\frac{dx}{ds} = \cos\theta$; therefore

$$dx = \frac{a}{\sqrt{2}} \frac{\cos\theta \, d\theta}{\sqrt{C - \cos\theta}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Substituting for C in terms of h from (18), we have

$$y = 2a\left(\frac{h^2}{4a^2} + \sin^2\frac{\theta}{2}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (20)$$

$$dx = \frac{a}{2} \cdot \frac{\cos\theta\, d\theta}{\left(\frac{h^2}{4a^2} + \sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The value of x can be expressed in terms of the ordinary elliptic integrals by putting $\theta = \pi - 2\phi$. If then we put $k^2 = \frac{4a^2}{4a^2 + h^2}$, so that k is $<$ 1, we have

$$dx = \frac{2a}{k}\left\{\frac{k^2 - 2}{2}\frac{d\phi}{\Delta(\phi)} + \Delta(\phi)\, d\phi\right\}, \quad . \quad . \quad (22)$$

where, as usual, $\Delta(\phi) \equiv \sqrt{1 - k^2 \sin^2\phi}$. The limits of θ being 0 and $\frac{\pi}{2} - i$, where i is the angle of contact, those of ϕ are $\frac{\pi}{2}$ and $\frac{\pi}{4} + \frac{i}{2}$. The figure supposes the angle of contact to be acute, as when water rises between two glass plates; if it is obtuse, as when mercury is depressed between two glass plates, the discussion proceeds in the same manner.

Two plates close together in a liquid move towards each other, as if by attraction, whether the liquid rises or is depressed between them—as was first explained by Laplace. In all such cases of approach between bodies floating on a liquid the result is due to an excess of pressure on their backs, or remote faces, over the pressure on their adjacent faces. Thus, on the plate OB above B the intensity of pressure is the same on both sides, being that of the atmosphere; also below O the intensities are the same, and again at both sides of OA; but between A and B the intensity of pressure on the left side is that of the atmosphere while on the right it is less. For if on the surface AB between A and B we take any point R, at a height z above Ox, the intensity of pressure exerted by the liquid on the plane is $p_0 - wz$, where p_0 = atmospheric intensity, since it has been shown in Art. 72 that the intrinsic pressure K disappears. Hence the total pressure on the plane AB from left to right is less than that from right to left; and similarly for $A'B'$; therefore the planes approach. The same result follows if (as in Fig. 36) the liquid is depressed between both planes.

But if the liquid rises in contact with one plane and is depressed in contact with the other, the two plates are urged away from each other.

Suppose the liquid to rise in contact with the plate AB (Fig. 45) and to fall in contact with the plate $A'B'$; then the level of the liquid at the left of the first must be higher than that at the right; and the depression of the liquid at the right of the second plate is greater than at the left. Evidently, then, the portion AB of the first plate experiences an excess of pressure towards the left, while, the pressure at the left of $B'A'$ being greater than that of the atmosphere, the second plate experiences an excess of pressure towards the right. Thus the plates move away from each other.

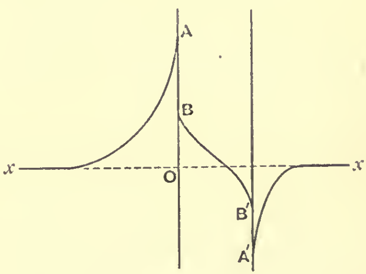

Fig. 45.

This can be shown experimentally by placing on the surface of water two small pieces of cork, one of which is clean and the other greased.

Next, taking the case in which a liquid rises inside a narrow vertical cylindrical tube, the free surface of the liquid inside the tube will be one of revolution; and if y is measured from the line Ox, which is the level of the plane surface, the equation of the surface of the liquid is

$$\frac{1}{\rho} + \frac{1}{n} = \frac{y}{a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where ρ is the radius of curvature of the meridian at any point, and n the length of the normal between this point and the axis of the tube.

If the horizontal line through the lowest point, V, of the meridian (Fig. 39) is taken as axis of x, the equation becomes

$$\frac{1}{\rho} + \frac{1}{n} = \frac{h+y}{a^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and if we put $\frac{d\theta}{ds}$ and $\frac{\sin\theta}{x}$ for $\frac{1}{\rho}$ and $\frac{1}{n}$, this becomes

$$\frac{1}{x}\frac{d}{dx}(x\sin\theta) = \frac{h+y}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

This equation cannot be integrated accurately; but an approximate solution can be obtained by the following method, which is, in principle, the same as that employed by Laplace (Supplement to Book X of the *Mécanique Céleste*).

Take a circle having its centre on the vertical through V and having a radius c; and let us determine this circle in such a way that its ordinate (for any abscissa) differs very little from the ordinate of the point, p, on the curve of the liquid which has the same abscissa. The ordinate of the circle is given by the equation

$$y = b - \sqrt{c^2 - x^2},$$

so that for any point on the curve we have

$$y = b - \sqrt{c^2 - x^2} + \zeta, \quad . \quad . \quad . \quad . \quad (26)$$

where ζ is a very small quantity. This gives

$$\frac{dy}{dx} = \frac{x}{\sqrt{c^2 - x^2}} + \frac{d\zeta}{dx}, \quad . \quad . \quad . \quad . \quad (27)$$

$$\therefore \ x \sin\theta = \frac{x^2}{c} + \frac{x(c^2 - x^2)^{\frac{3}{2}}}{c^3} \frac{d\zeta}{dx}, \quad . \quad . \quad . \quad (28)$$

neglecting the square of $\dfrac{d\zeta}{dx}$. Hence (25) becomes

$$\frac{d}{dx}\left\{\frac{x^2}{c} + \frac{x(c^2 - x^2)^{\frac{3}{2}}}{c^3} \frac{d\zeta}{dx}\right\} = \frac{hx + xy}{a^2}$$

$$= \frac{h + b - \sqrt{c^2 - x^2}}{a^2} . x, \quad (29)$$

by neglecting ζ in the value of y. Integrating between $x = 0$ and $x = x$,

$$\frac{x^2}{c} + \frac{x(c^2 - x^2)^{\frac{3}{2}}}{c^3} \frac{d\zeta}{dx} = \frac{(h + b)\, x^2}{2a^2} + \frac{(c^2 - x^2)^{\frac{3}{2}}}{3a^2} - \frac{c^3}{3a^2}, \quad (30)$$

$$\therefore \ \frac{d\zeta}{dx} = \left(\frac{h + b}{2a^2} - \frac{1}{c}\right) \frac{c^3 x}{(c^2 - x^2)^{\frac{3}{2}}}$$

$$+ \frac{c^3}{3a^2 x} - \frac{c^6}{3a^2} \cdot \frac{1}{x(c^2 - x^2)^{\frac{3}{2}}}, \quad (31)$$

$$\therefore\ \zeta = \left(\frac{h+b}{2a^2} - \frac{1}{c} - \frac{c}{3a^2}\right)\frac{c}{\sqrt{c^2-x^2}}$$

$$+\frac{c^3}{3a^2}\log_e\left(c+\sqrt{c^2-x^2}\right)+C, \quad (32)$$

where C is a constant to be determined.

Now this equation would make $\zeta = \infty$ when $x = c$, which would, of course, be absurd. Hence we must have

$$\frac{h+b}{2a^2} - \frac{1}{c} - \frac{c}{3a^2} = 0. \quad . \quad . \quad . \quad . \quad (33)$$

Again, $y = 0$ when $x = 0$, $\therefore\ b - c + \zeta = 0$ at V, and (32) becomes

$$\zeta = c - b + \frac{c^3}{3a^2}\log_e\frac{c+\sqrt{c^2-x^2}}{2c}, \quad . \quad . \quad . \quad (34)$$

so that from (26)

$$y = c - \sqrt{c^2-x^2} + \frac{c^3}{3a^2}\log_e\frac{c+\sqrt{c^2-x^2}}{2c}, \quad . \quad (35)$$

which is the approximate equation of the curve when c is known. Now we know that at the points B, B', of contact with the tube $\frac{dy}{dx} = \cot i$, and therefore if r is the radius of the tube,

$$\cot i = \frac{r}{\sqrt{c^2-r^2}}\left\{1 - \frac{c^3\left(c-\sqrt{c^2-r^2}\right)}{3a^2r^2}\right\} \quad . \quad . \quad (36)$$

which determines c; and b is then known from (33).

As a first approximation, (36) gives $c = r\sec i$ from which, more accurately,

$$c = r\sec i\left\{1 - \frac{r^2}{3a^2}\frac{\sin^2 i\,(1-\sin i)}{\cos^4 i}\right\}. \quad . \quad . \quad (37)$$

Finally, take the case in which liquid is contained between two vertical plates which make with each other a very small angle. Let the plates be $AyOx$, $A'yOx'$ (Fig. 46), intersecting in the vertical line Oy, and making with each other the very small angle ϵ, or xOx'. Let the curves in which the liquid surface intersects the plates be $yPQx$, $yP'Q'x'$. It is required to find the nature of these curves.

Take any two indefinitely near points, P, Q, on one curve; let the corresponding points on the other be P', Q', the lines PP', QQ' being normal to the plates and in the liquid surface; draw the ordinates Pm, Qn, &c., and consider the separate equilibrium of the small prism $PQ'm$.

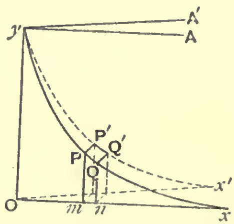

Fig. 46.

If $Pm = y$, $Om = x$, the weight of this prism is $\epsilon wxydx$, where $mn = dx$, and it is balanced by surface tension round the contour $PQQ'P'$. Let T be the surface tension, and θ the inclination of the tangent to the curve PQ at P to Ox.

Then the amount of tension on PP' is $T \cdot \epsilon x$, and its vertical component is $\epsilon Tx \sin\theta$; therefore the vertical component of the tension on QQ' is

$$\epsilon Tx \sin\theta + \epsilon T \frac{d\,(x \sin\theta)}{dx}\, dx.$$

Also the tension on PQ acts in the tangent plane to the liquid surface and at right angles to the line PQ. Now if i is the angle of contact (i. e., the angle between this tangent plane and the plate yOx), we easily find that the direction-cosines of the tangent plane are proportional to $\sin\theta$, $\cos\theta$, $\cot i$, while those of the line PQ are $\cos\theta$,

$-\sin\theta$, 0. Hence the direction-cosines of the line of action of the tension on PQ are $\sin\theta\cos i$, $\cos\theta\cos i$, $\sin i$; and if $PQ = ds$, the vertical component of the tension is $Tds\,.\cos\theta\cos i$, or $T\cos i\,.\,dx$; while the tension on $P'Q'$ gives the same amount. Hence for the equilibrium of the prism we have

$$2T\cos i = \epsilon wxy + \epsilon T\frac{d\,(x\sin\theta)}{dx}, \quad . \quad . \quad . \quad (38)$$

which shows that the second term on the right-hand side is of the order ϵ^2.

Neglecting it in comparison with the first, we have for the approximate equation of the curve

$$xy = \frac{2T\cos i}{\epsilon w} = \frac{2a^2\cos i}{\epsilon}; \quad . \quad . \quad . \quad . \quad (39)$$

where, as previously, $a^2 = \dfrac{T}{w}$.

The curve is, then, approximately a rectangular hyperbola —a result which is commonly assumed in virtue of the fact that the elevation of a liquid between two parallel close plates varies inversely as the distance between them.

83. Liquid Films. The forms which can be assumed by the surface of a liquid which is under the influence of none but molecular forces can be produced by means of thin films of liquid, such as soap-bubbles. Imagine a thin film of liquid in contact with air at both sides of its surface, the intensity of pressure of the air being, in general, different on these sides.

Let $ABCD$, Fig. 47, be a portion of such a film; let P be any point on its surface; let PQ, PS be elements of the arcs of the two principal sections of the surface at P; at Q and S draw the two principal sections QR and SR. Thus we determine a small area $PQRS$ on the surface.

Let the normals to the surface at P and S intersect in C_2, and those at P and Q in C_1. Then $PC_1 = R_1$, $PC_2 = R_2$, where R_1, R_2 are the principal radii of curvature of the surface at P.

Let $PQ = ds_1$, $PS = ds_2$, p = intensity of air pressure on the lower or concave side of the surface at P, and p_0 = intensity of air pressure on the convex side. Then $(p-p_0)\, ds_1 ds_2$ is the resultant air pressure on the area $PQRS$ in the sense C_1P; and for the equilibrium of the element this must be equal to the component of force in the sense PC_1, given by the surface tension exerted on the contour of the element,

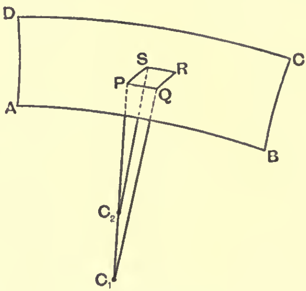

Fig. 47.

assuming that the film is so thin that the action of gravity is negligible. Now if T is the tension per unit length along PS, the whole tension on PS (which acts at its middle point perpendicularly to PS) is $T \,.\, ds_2$, and the component of this along the normal to the surface is $Tds_2 \,.\, \sin \dfrac{PC_1Q}{2}$, or $Tds_2 \,.\, \dfrac{ds_1}{2R_1}$. The tension on QR gives a component of the same magnitude; hence the sum of these is $\dfrac{T}{R_1} ds_1 ds_2$; similarly the sum of the normal components of the tensions acting on the sides PQ and SR is $\dfrac{T}{R_2} ds_1 ds_2$; so that the normal component of the tension acting on the whole contour $PQRS$ is

$$T\left(\frac{1}{R_1} + \frac{1}{R_2}\right) ds_1 ds_2. \quad \ldots \ldots \quad (1)$$

If the thickness of the film exceeds 2ϵ, where ϵ is the radius of molecular activity, there will be surface tension exerted both at the upper and at the under side of the surface, this action being confined (as explained in Art. 78) to a layer of thickness ϵ at each of these sides; so that we must replace T in (1) by $2T$, and the equation of equilibrium is

$$2T\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=p-p_0. \quad . \quad . \quad . \quad . \quad (2)$$

Hence, since p and p_0 are the same at points of the film, the equation of its surface is

$$\frac{1}{R_1}+\frac{1}{R_2}=\text{const.}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the forms of these films are the same as those of the surface of a liquid which is not acted upon by any external force, i. e., the shapes of thin films are the same as those of drops of oil in the water-alcohol mixture of Plateau (see p. 162).

The equation (2) can be otherwise deduced without considering the separate equilibrium of an element of the film. For, the intensity of pressure at any point inside the film (beyond the depth ϵ) due to the convex side is $p_0+K+T\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$; and the intensity of pressure at the same point due to the inner, or concave, side is

$$p+K-T\left(\frac{1}{R_1}+\frac{1}{R_2}\right).$$

Equating these, we have (2).

For a spherical bubble $R_1=R_2=r$,

$$\therefore\ 4T=(p-p_0)r, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which shows that for all sizes of the bubbles the product $(p-p_0)\,r$ remains constant.

One possible shape of the bubble is that of a cylinder closed by two spherical ends. If r is the radius of the cylinder, r' that of each end, we have

$$\frac{2T}{r} = p - p_0,$$

$$\frac{4T}{r'} = p - p_0,$$

$$\therefore \quad r' = 2r.$$

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